Cardinality of Joint and Disjoint Sets and a Related Discrete Distribution

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ABSTRACT

The number of elements belonging to the intersection of some overlapped sets and that belonging precisely to the junction of some overlapped sets are related. We have demonstrated a more straightforward presentation with some examples for a broad spectrum of students and instructors. It shows an apparent relationship between the cardinality of joint and disjoint sets. There is a probability distribution associated with it, and it is unexplored. We have provided some examples to illustrate it. We have derived the factorial moment structure of the probability distribution for the first time, and they found it to be elegant. We have also derived raw and corrected moments of the distribution.

Key words:
Cardinality, Joint Sets, Disjoint Sets, Discrete Distribution, Probability Distribution, Factorial Moment Structure

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INTRODUCTION

Many real-world problems involve the intersection of finite sets, and one needs to find the number of elements belonging only (exactly) to single sets, the intersection of two sets, the intersection of three sets, etc. (Kolman et al., 1992, pp198-200). It is popular to use the Venn Diagram for this kind of problem. The cardinality of a set is a measure of the “number of elements” of the set. For example, the set $C=\{1,\ 9,\ 7,\ 1\}$ has four elements, and hence, the cardinality of the set is $N(C)=4$.

However, if the number of sets (k) overlapped in the problem is four or more, determining the number of elements belonging precisely to the intersection of some sets becomes a formidable job. The interested reader is referred to Björklund, Husfeldt, and Koivisto (2009) for a modern set partitioning method. In this paper, we delineate a very straightforward method for any number of sets by illustrating the idea with some examples. This sheds light on the relationship between the cardinality between joint (or overlapped sets) and disjoint (mutually exclusive) sets. The general relationship is known in combinatorics but not popularly known. This prompted us to make a milder presentation for pedagogical purposes, though the general theorem is presented in Theorem 2.1.

Counting Techniques are needed to count, enumerate, or find possible solutions to many real-world problems. The application is overwhelming in computer science, engineering, natural and social science, biomedicine, operations research, business, etc. See, for example, Mycielski et al. (1997). The generation of combinatorial sequences, such as permutations and combinations, has been studied extensively because of the fundamental nature and the importance of practical applications. There has been interest in the generation of these sequences in a parallel or distributed computing environment (Akl, 1989; Kapralski, 1993)

A milestone example in physics is the number of ways of distributing some particles among some sublevels of an energy level. The combinatorial result later yielded a Bose-Einstein distribution. With the spirit of converting combinatorial results into a discrete probability distribution, we define a discrete probability distribution and find its elegant moment structure.

Suppose that there are k sets $A_1, A_2, ..., A_k$ that may be overlapping. Define $N(A)$ as the number of elements belonging to a set A_j, $(j = 1, 2, ..., k)$. Then $k = 2$ we define $n_{(1)} = N(A_1) + N(A_2)$ the number of components in any of the single sets, $n_{(2)} = N(A_1A_2)$ and the number of elements in a 2-combinations intersecting set. We also define $n_1 = N(A_1A_2') + N(A_1'A_2)$, the number of elements in precisely one of the single sets. Note that A_1A_2' $A_1'A_2$ they are disjoint or mutually exclusive sets. Also, it $n_2 = N(A_1A_2)$ is the number of elements belonging exactly to a 2-combinations intersecting set. Obviously $n_2 = n_{(2)}$ and $n_1 = N(A_1A_2') + N(A_1'A_2) = N(A_1) - N(A_1A_2) + N(A_2) - N(A_1A_2) = n_{(1)} - 2n_{(2)}$.

In fact, for k overlapping sets, $n_{(k)}$ they n_k are the same.

Example 1.1 Suppose that 60 candidates appeared for language and math tests for a job.

Thirty of them passed in language, 25 of them passed in math, and 5 of them passed in both language and math. What number of candidates passed only one of the above two subjects?

Solution: $k = 2,$ We define $n_{(1)} = N(A_1) + N(A_2),$ the number of candidates passing any single subject $n_{(2)} = N(A_1A_2)$ and the number passing both subjects. We also define $n_1 = N(A_1A_2') + N(A_1'A_2),$ the number of candidates passing only one subject. $n_{(1)} = N(A_1) + N(A_2) = 30 + 25 = 55,$ $n_{(2)} = 5.$ Also, by definition, we have

$n_1 = N(A_1A_2') + N(A_1A_2) = N(A_1) - N(A_1A_2) + N(A_2) - N(A_1A_2),$ which can be simplified to

$n_1 = N(A_1) + N(A_2) - 2N(A_1A_2)$. Thus, we have proved $n_1 = n_{(1)} - 2n_{(2)}$. For the example, $n_1 = 55 - 2(5) = 45$.

For $k = 3$, $n_{(1)}$ is the sum of the cardinalities of three joint or overlapped sets, namely,

A_1, A_2, A_3, However, it n_1 is the sum of the cardinalities of three disjoint or mutually exclusive sets; namely, $A_1A_2'A_3'$, $A_1'A_2A_3'$, $A_1'A_2'A_3$. Thus we define $n_{(1)} = N(A_1) + N(A_2) + N(A_3)$, the number of elements in any of the single sets, $n_{(2)} = N(A_1A_2) + N(A_1A_3) + N(A_2A_3)$, the number of elements in any of the 2-combinations intersecting sets formed from the three sets, $n_{(3)} = N(A_1A_2A_3)$, the number of elements in the intersection of three sets.

We also define $n_1 = N(A_1A_2'A_3') + N(A_1'A_2A_3') + N(A_1'A_2'A_3)$, the number of elements belonging only (exactly) to one of the 3 sets, $n_2 = N(A_1A_2A_3') + N(A_1A_2'A_3) + N(A_1'A_2A_3)$, the number of elements belonging only (exactly) to any of the 2-combinations intersecting sets formed from the three sets, $n_3 = N(A_1A_2A_3)$, the number of elements belonging exactly to the intersection of three sets. Note that each of n_1 n_2, them has a sum of numbers (cardinality) of 3 disjoint or mutually exclusive sets.

Similar to the case, $k = 2$, we can prove the following identities:

$$n_1 = n_{(1)} - 2n_{(2)} + 3n_{(3)}, \tag{1.1}$$

$$n_2 = n_{(2)} - 3n_{(3)}, \tag{1.2}$$

$$n_3 = n_{(3)}. \tag{1.3}$$

The third identity is apparent. To prove the identity (1.2), we proceed as follows:

The definition of n_2 the above can be spelled out as follows:

$$n_2 = N(A_1A_2) - N(A_1A_2A_3) + N(A_1A_3) - N(A_1A_2A_3) + N(A_2A_3) - N(A_1A_2A_3),$$

$$n_2 = N(A_1A_2) + N(A_1A_3) + N(A_2A_3) - 3n_{(3)}.$$

Since the first three terms on the right-hand side of the above identity are the same as $n_{(2)}$, we have proved the identity (1.2) above. Next, consider the following:

$$n_{(1)} - n_{(2)} = N(A_1) + N(A_2) + N(A_3) - n_{(2)},$$

$$n_{(1)} - n_{(2)} = \{N(A_1) - N(A_1A_2)\} + \{N(A_2) - N(A_2A_3)\} + \{N(A_3) - N(A_1A_3)\},$$

$$n_{(1)} - n_{(2)} = \{N(A_1A_2'A_3') + N(A_1A_2'A_3)\} + N(A_1A_2A_3') + N(A_1'A_2A_3') + \{N(A_1'A_2A_3) + N(A_1'A_2'A_3)\},$$

$$n_{(1)} - n_{(2)} = n_1 + n_2.$$

Since we already $n_2 = n_{(2)} - 3n_{(3)}$, established, the identity (1.1) is proved.

Example 1.2 One hundred workers are grouped by their areas of expertise and are placed on at least one team. Forty are on the marketing team (A_1), , 30 are on the Sales team (A_2), , and 20 are on the Vision team (A_3).. Nine workers are on both the Marketing and Sales teams (A_1A_2), , five are on both the Marketing and Vision teams (A_1A_3), , six are on both the Sales and Vision teams (A_2A_3), , and four are on all three teams $(A_1A_2A_3)$. How many workers were not assigned to any of the three teams?

Solution: From the problem, we have the following:

$$n_{(1)} = N(A_1) + N(A_2) + N(A_3) = 40 + 30 + 20 = 90,$$

$$n_{(2)} = N(A_1A_2) + N(A_1A_3) + N(A_2A_3) = 9 + 5 + 6 = 20,$$

$$n_{(3)} = N(A_1A_2A_3) = 4.$$

Then, n_1, n_2 and n_3 can be easily determined by the following relations:

$$n_3 = n_{(3)} = 4,$$

$$n_2 = n_{(2)} - 3\,n_{(3)} = 20 - 3(4) = 8,$$

$$n_1 = n_{(1)} - 2n_{(2)} + 3n_{(3)} = 90 - 2(20) + 3(4) = 62,$$

$$n_0 = n - n_1 - n_2 - n_3 = 100 - (62 + 8 + 4) = 26.$$

Twenty-six workers were assigned to only some of the three teams.

We also mention that $n_1 + n_2 + n_3 = n_{(1)} - n_{(2)} + n_{(3)}\ n_2 + n_3 = n_{(1)} - 2n_{(3)}$.By using De Morgan's Law, one can solve the problem as follows:

$$N(A_1'A_2'A_3') = N(A_1 \cup A_2 \cup A_3)' = 1 - N(A_1 \cup A_2 \cup A_3)$$

where

$$N(A_1 \cup A_2 \cup A_3) = N(A_1) + N(A_2) + N(A_3) - [N(A_1A_2) + N(A_1A_3) + N(A_2A_3)] + N(A_1A_2A_3),$$

$$N(A_1 \cup A_2 \cup A_3) = 40 + 30 + 20 - (9 + 5 + 6) + 4 = 74.$$

Finally, $N(A_1'A_2'A_3') = 1 - 74 = 26.$

In passing, we provide formulae for four overlapping sets as follows:

$$n_4 = n_{(4)},$$

$$n_3 = n_{(3)} - 4\, n_{(4)},$$

$$n_2 = n_{(2)} - 3\, n_{(3)} + 6n_{(4)},$$

$$n_1 = n_{(1)} - 2n_{(2)} + 3n_{(3)} - 4n_{(4)},$$

$$n_0 = n - n_1 - n_2 - n_3 - n_4.$$

Similarly, we can define formulae for k sets. We define

$$n_{(1)} = N(A_1) + N(A_2) + N(A_3) + \ldots + N(A_k),$$
$$n_{(2)} = N(A_1A_2) + N(A_1A_3) + N(A_2A_3) + \ldots + N(A_{k-1}A_k),$$

$$n_{(3)} = N(A_1A_2A_3) + N(A_1A_2A_4) + \ldots + N(A_{k-2}A_{k-1}A_k), \ldots$$

And so on. We also define

$$n_1 = N(A_1A_2'A_3'\ldots A_k') + N(A_1'A_2A_3'\ldots A_k') + \ldots + N(A_1'A_2'\ldots A_{k-1}'A_k),$$

$$n_2 = N(A_1A_2A_3'\ldots A_k') + N(A_1A_2'A_3A_4'\ldots A_k') + \ldots + N(A_1'A_2'\ldots A_{k-2}'A_{k-1}A_k),$$

$$n_3 = N(A_1A_2A_3A_4'\ldots A_k') + N(A_1A_2A_3'A_4A_5'\ldots A_k') + \ldots + N(A_1'A_2'\ldots A_{k-3}'A_{k-2}A_{k-1}A_k), \ldots$$

And so on. The following theorem is well known (Eisen, 1969, p113).

Theorem 1.1 Suppose that there are k sets $A_1, A_2, \ldots, A_k$. Define $N(A)$ the number of elements belonging to a set A. Also, define n_i = the number of elements belonging precisely to any of the i-combinations $(i = 1, 2, \ldots, k)$ intersecting sets $n_{(j)}$ = and the number of

components in any of the j-combinations $(j = 1,2,\ldots,k)$ intersecting sets. Then, the following relationship exists between n_i $(i = 1, 2, \ldots, k)$ and $n_{(j)}$ $(j = 1, 2, \ldots, k)$.

$$n_i = n_{(i)} - \binom{i+1}{i} n_{(i+1)} + \binom{i+2}{i} n_{(i+2)} - \ldots + (-1)^{k-i} \binom{k}{i} n_{(k)}, \quad (i = 1,\ 2,\ \ldots, k). \tag{1.4}$$

Many problems of discrete mathematics (see, e.g., Kolman, Anton, and Averbach, 1992, pp. 198- 200) can be solved by the above theorem to have better insight.

Eisen (1969, p156) described that the above theorem can be used as a probability distribution (See Theorem 2.1 below). Still, we have not found this probability distribution used for solving problems in statistics books, nor is there any attempt to study the distribution further. The formidable calculation in deriving the moments of the distribution is made much easier and neater by introducing a matrix and exploiting its property. In this paper, we derive the factorial moment structure of the probability distribution and find it to be exquisite. Raw and corrected moments of the distribution are also derived.

Probability Distribution of the Number of Elements Belonging Exactly to the Intersection of Some Sets

The relationship between $\pi_i = n_i / n,\ (i = 1, 2, ..., k)$ and $\pi_{(j)} = n_{(j)} / n$, $(j = 1, 2, ..., k)$, where $n = n_0 + n_1 + \cdots + n_k$, is given by the following theorem.

Theorem 2.1 For any integer i with $0 \le i \le k$, the probability π_i that precisely i among k events $A_1, A_2, ..., A_k$ coincide is given by

$$\pi_i = \pi_{(i)} - \binom{i+1}{i}\pi_{(i+1)} + \binom{i+2}{i}\pi_{(i+2)} - ... + (-1)^{k-i}\binom{k}{i}\pi_{(k)}, \quad (i = 1, 2, ..., k). \tag{2.1}$$

where $\pi_{(0)} = 1$, $\pi_{(1)} = \sum P(A_i)$, $\pi_{(2)} = \sum P(A_i A_j), \cdots,$ $\pi_{(k)} = P(A_1 A_2 \cdots A_k)$ (cf. Eisen,1969, 156).

Example 2.1 It is found that 36% of people in a city read the newspaper A_1, 27% of them read the newspaper A_2, 3% of them read both the newspapers A_1 and A_2. Then $\pi_{(2)} = P(A_1A_2) = 0.03,\quad \pi_{(1)} = P(A_1) + P(A_2) = 0.36 + 0.27 = 0.63$, consequently, the probability that a person reads exactly two newspapers is given by $\pi_2 = \pi_{(2)} = 0.03$, the probability that a person reads exactly one newspaper is provided by $\pi_1 = \pi_{(1)} - 2\pi_{(2)} = 0.63 - 2(0.3) = 0.57$ and the probability that a person does not read either of the two newspapers $\pi_0 = 1 - 0.57 - 0.03 = 0.40$.

Example 2.2 Suppose that 36 % of the people in a community read a newspaper A_1, 27% of them read the newspaper A_2, 22% of them read the newspaper A_3, 3% of them read both the newspapers A_1 and A_2, 4% of them read both the newspapers A_1 and A_3, 5% of them read both the newspapers A_2 and A_3, and only 1% of them read all the newspapers.

Let the probability that a randomly selected person read exactly one newspaper, exactly two newspapers, and all three newspapers be denoted π_1, π_2 and π_3, respectively. Since $k = 3$, it follows from Theorem 2.1 that

$$\pi_{(3)} = P(A_1A_2A_3) = 0.01,$$

$$\pi_{(2)} = P(A_1A_2) + P(A_1A_3) + P(A_2A_3) = 0.03 + 0.04 + 0.05 = 0.12,$$

$$\pi_{(1)} = P(A_1) + P(A_2) + P(A_3) = 0.36 + 0.27 + 0.22 = 0.85.$$

Then by (2.1) we have

$$\pi_3 = \pi_{(3)} = 0.01,$$

$$\pi_2 = \pi_{(2)} - 3\ \pi_{(3)} = 0.12 - 3(0.01) = 0.09,$$

$$\pi_1 = \pi_{(1)} - 2\pi_{(2)} + 3\pi_{(3)} = 0.85 - 2(0.12) + 3(0.01) = 0.64,$$

$$\pi_0 = 1 - \pi_1 - \pi_2 - \pi_3 = 0.26.$$

Note that $\pi_{(1)}, \pi_{(2)}, \pi_{(3)}$ they do not constitute a genuine set of probabilities but π_3, π_2, π_1 and π_0 do.

Example 2.3 It is found that 38% of people in a city read the newspaper A_1, 49% of them read the newspaper A_2, 43% of them read the newspaper A_3, and 33% of them read the newspaper A_4. 22% of them read both newspapers A_1 A_2, 11% of them read both newspapers A_1 A_3, 11% of them also read both newspapers A_1 A_4, 22% of them read both newspapers A_2 A_3, 12% of them read both newspapers A_2 and A_4 and 14% of them read both newspapers A_3 and A_4. 8% of them read the newspapers $A_{1,}A_2$ and A_3, 6% of them read the newspapers $A_{1,}A_2$ and A_4, 4% of them read the newspapers $A_{1,}A_3$ and A_4, 7% of them read the newspapers $A_{2,}A_3$ and A_4, and only 1% reads all the four newspapers.

To find π_i, $(i = 1, 2, 3, 4)$, the proportion of people who read precisely $i(i = 1,2,3,4)$ newspapers, we proceed as follows.

$$\pi_{(4)} = P(A_1A_2A_3A_4) = 0.01,$$

$$\pi_{(3)} = P(A_1A_2A_3) + P(A_1A_2A_4) + P(A_1A_3A_4) + P(A_2A_3A_4)$$
$$= 0.08 + 0.06 + 0.04 + 0.07 = 0.25,$$

$$\pi_{(2)} = P(A_1A_2) + P(A_1A_3) + P(A_1A_4) + P(A_2A_3) + P(A_2A_4) + P(A_3A_4)$$
$$= 0.22 + 0.11 + 0.11 + 0.22 + 0.12 + 0.14 = 0.92,$$

$$\pi_{(1)} = P(A_1) + P(A_2) + P(A_3) + P(A_4) = 0.38 + 0.49 + 0.43 + 0.33 = 1.63.$$

Then by (2.1) we have

$$\pi_4 = \pi_{(4)} = 0.01,$$

$$\pi_3 = \pi_{(3)} - 4\pi_{(4)} = 0.25 - 4(0.01) = 0.21,$$

$$\pi_2 = \pi_{(2)} - 3\pi_{(3)} + 6\pi_{(4)} = 0.92 - 3(0.25) + 6(0.1) = 0.23,$$

$$\pi_1 = \pi_{(1)} - 2\pi_{(2)} + 3\pi_{(3)} - 4\pi_{(4)} = 1.63 - 2(0.92) + 3(0.25) - 4(0.01) = 0.50,$$

$$\pi_0 = 1 - (\pi_1 + \pi_2 + \pi_3 + \pi_4) = 1 - (0.50 + 0.23 + 0.21 + 0.01) = 0.05.$$

Many problems of elementary probability (e.g., # 2.4 in Hines and Montgomery, 1990, pp.57-58) can be solved by Theorem 2.1 to have better insight.

Moments of the Distribution

Let us now calculate the moment a probability distribution is discussed in the following example (which is based on Example 2.2).

Example 3.1 Let X = several newspapers read by a person. Then the probability density function X is given by $f(X = i) = \pi_i = n_i / n, \ (i = 0, 1, 2, 3)$ i.e.

$f(0) = 0.26, \ f(1) = 0.64, \ f(2) = 0.09, \ f(3) = 0.01$. Then, the expected number of newspapers read by a person is given $E(X) = \sum_{i=0}^{3} i \ f(i) = 0.85$. The second raw moment is provided by $E\left(X^2\right) = \sum_{i=0}^{3} i^2 f(i) = 1.09$, and hence the variance of X is provided by $V(X) = E\left(X^2\right) - \left[E(X)\right]^2 = 0.3675$.

The Factorial Moments of $X \ k (\geq 2)$ Sets

Let X be a random variable with the following probability density function:

$$P(X = i) = \pi_i = n_i / n \ , \ i = 0, 1, ..., k \qquad (3.1)$$

Where n_i is the number of elements belonging precisely to the intersection of some sets defined in (1.4). In k a large case, it would be easier to calculate factorial moments, which can be used to calculate raw moments and, hence, corrected moments. In what follows, we introduce a matrix that makes the algebra of finding the factorial moments of the above distribution neater. The formula in (1.4) can be written as

$$n_i = \sum_{i=1}^{k} \sum_{j=1}^{k} (-1)^{j-i} \binom{j}{i} n_{(j)} \ , \quad (i = 1, 2, ..., k)$$

or simply by, $\underset{\sim}{n} = C \underset{\sim}{n}_{(.)} = C\underset{\sim}{m}$ (3.2) where $\underset{\sim}{n}' = (n_1, n_2, \cdots, n_k)$, $C = ((c_{ij}))$, $c_{ij} = (-1)^{j-i}\binom{j}{i}$, $i = 1,2,\ldots,k;\ j = 1,2,\ldots,k$ with $c_{ij} = 0$ if, $i > j$ and one if $i = j$. Further, let $n_{(i)} = m_i$, $(i = 1,2,\ldots,k)$ so that $\underset{\sim}{m}' = (m_1, m_2, \ldots, m_k)$. The matrix C can then be written as

$$C = \begin{bmatrix} 1 & -2 & +3 & -4 & \ldots & (-1)^{k-2}(k-1) & (-1)^{k-1}k \\ 0 & 1 & -3 & 6 & \ldots & (-1)^{k-3}\binom{k-1}{2} & (-1)^{k-2}\binom{k}{2} \\ 0 & 0 & 1 & -4 & \ldots & (-1)^{k-4}\binom{k-1}{3} & (-1)^{k-3}\binom{k}{3} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & (-1)^{-1}\binom{k}{k-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $C(l,l),\ (l = 1,2,\ldots,k)$ be the principal sub-matrix of order l i.e. a matrix formed C by deleting the first l rows and the first l columns. For example, $C(2,2)$ it is a matrix formed C by deleting the first two rows and the first two columns. Similarly $\underset{\sim}{m}(l)$, it is a vector created $\underline{m}$ after deleting the first l columns.

Lemma 3.1 With the above notations, $r = 1,2,\cdots,k$ the following holds:

$$\begin{pmatrix} \frac{r!}{0!} & \frac{(r+1)!}{1!} & \cdots & \frac{k!}{(k-r)!} \end{pmatrix} C(r-1, r-1) = (r!\ 0\ 0\ \cdots\ 0).$$

Proof. The proof is straightforward.

In what follows, we derive factorial moments X using Lemma 3.1.

Theorem 3.1. The r th factorial moment of the distribution X is given by

$$\mu'_{(r)} = E\left[X^{(r)}\right] = \begin{cases} r!\ \dfrac{n_{(r)}}{n} = r!\pi_{(r)}, & r \le k. \\ 0, & r \ge k+1. \end{cases} \tag{3.3}$$

Proof.

$$E[X^{(1)}]=\sum_{i=0}^{k} i\ \ (n_i/n)=(1\ \ 2\ \ 3\ \ \ldots\ \ k)\ \underset{\sim}{n}/n$$
$$=(1\ \ 2\ \ 3\ \ \ldots\ \ k)\ \ C\ \ \underset{\sim}{m}/n=\ (1\ \ 0\ \ \ \ldots\ \ 0)\ \ \underset{\sim}{m}/n$$
$$=m_2/n=n_{(1)}/n=\pi_{(1)},$$

$$E[X^{(2)}]=\sum_{i=0}^{k} i(i-1)(n_i/n)$$
$$=[(2)(1)n_2+)(3)(2)n_3+(4)(3)n_3+\ldots+k(k-1)n_k]/n$$
$$=\begin{bmatrix}\frac{2!}{0!} & \frac{3!}{1!} & \frac{4!}{2!}\ldots & \frac{k!}{(k-2)!}\end{bmatrix}\ \underset{\sim}{n}(1)/n$$
$$=\begin{bmatrix}\frac{2!}{0!} & \frac{3!}{1!} & \frac{4!}{2!}\ldots & \frac{k!}{(k-2)!}\end{bmatrix}\ C(1,1)\underset{\sim}{m}(1)/n$$
$$=(2!\ \ 0\ \ 0\ \ldots\ 0)\ \ \underset{\sim}{m}(1)/n$$
$$=\ 2!m_2/n=\ \ 2!n_{(2)}/n=2!\ \ \pi_{(2)},$$

and

$$E[X^{(3)}]=\sum_{i=0}^{k} i(i-1)(i-2)(n_i/n)$$
$$=[(2)(1)n_2+)(3)(2)n_3+(4)(3)n_3+\ldots+k(k-1)n_k]/n$$
$$=\begin{bmatrix}\frac{3!}{0!} & \frac{4!}{1!} & \frac{5!}{2!}\ldots & \frac{k!}{(k-3)!}\end{bmatrix}\underset{\sim}{n}(2)/n$$
$$=\begin{bmatrix}\frac{3!}{0!} & \frac{4!}{1!} & \frac{5!}{2!}\ldots & \frac{k!}{(k-3)!}\end{bmatrix}C(2,2)\underset{\sim}{m}\ (2)/n$$
$$=(3!\ \ 0\ \ 0\ \ldots\ \ 0)\ \underset{\sim}{m}(2)/n$$
$$=\ \ 3!m_3/n=3!n_{(3)}/n=3!\ \ \pi_{(3)}.$$

In general $r\le k$, we have

$$E[X^{(r)}]=\sum_{i=0}^{k} i(i-1)(i-2)\ldots(i-r+1)(n_i/n)\ \ =\sum_{i=0}^{k}\frac{i!n_i}{(i-r)!n}$$
$$=\frac{1}{n}\left[\frac{r!}{0!}\ \ n_r+\frac{(r+1)!}{1!}\ \ n_{r+1}+\cdots+\frac{k!}{(k-r)!}\ \ n_k\right]$$
$$=\begin{bmatrix}\frac{r!}{0!} & \frac{(r+1)!}{1!} & \frac{(r+2)!}{2!}\ldots & \frac{k!}{(k-r)!}\end{bmatrix}\ n(r-1)/n,$$

Which can be written as

$$E[X^{(r)}]=\frac{1}{n}\begin{bmatrix}\frac{r!}{0!} & \frac{(r+1)!}{1!} & \frac{(r+2)!}{2!}\ldots & \frac{k!}{(k-r)!}\end{bmatrix} C(r-1,r-1)\underset{\sim}{m}(r-1)$$

$$=(r!\ \ 0\ \ 0\ \ \cdots\ \ 0)\ \underset{\sim}{m}\ (r-1)/n$$

$$=\ r!\ n_{(r)}/n=r!\ \pi_{(r)}.$$

The Moments of X for $k=1,\ 2,\ 3,\ 4$

The raw moments X are given by

$$\mu_r'=\sum_{i=1}^{r}S(r,i)\ \mu_{(i)}'=S(r,1)\mu_{(1)}'+S(r,2)\mu_{(2)}'+\ldots+S(r,r)\mu_{(r)}'$$

Where $S(r,i)$ is the Stirling number of the second kind (see Johnson et al., 1993, 44)? Simplicity $\mu_{(1)}'$ is traditionally denoted by μ . In particular, we have

$$\begin{aligned}\mu_1'&=\mu_{(1)}'=\mu,\\ \mu_2'&=\mu+\ \ \mu_{(2)}',\\ \mu_3'&=\mu'+3\mu_{(2)}'+\mu_{(3)}',\\ \mu_4'&=\mu_4'-7\mu_{(2)}'+6\mu_{(3)}'+\mu_{(4)}'.\end{aligned}\tag{3.4}$$

The corrected moments X can be calculated by the raw moments by the following well-known relations:

$$\begin{aligned}\mu_2&=\mu_2'-\mu^2,\\ \mu_3&=\mu_3'-3\mu_2'\mu+2\mu^3,\\ \mu_4&=\mu_4'-4\mu_3'\mu+6\mu_2'\mu^2-3\mu^4.\end{aligned}\tag{3.5}$$

It follows from (3.3) and (3.4) that the raw moments of X for $k=2$ are given by

$$\begin{aligned}\mu_1'(k=2)&=\mu_{(1)}'=\mu=\pi_{(1)},\\ \mu_2'(k=2)&=\ \mu+\mu_{(2)}'=\pi_{(1)}+2!\ \pi_{(2)}=\pi_{(1)}+2\pi_{(2)},\\ \mu_3'(k=2)&=\pi_{(1)}+\ 3\ \left(2!\ \ \pi_{(2)}\right)+0=\pi_{(1)}+6\pi_{(2)},\\ \mu_4'(k=2)&=\ \mu+7\ \mu_{(2)}'+0+0=\pi_{(1)}+7\left(2!\ \pi_{(2)}\right)=\pi_{(1)}+14\pi_{(2)}.\end{aligned}\tag{3.6}$$

Which can be presented as

$$\begin{pmatrix} \mu_1'(2) \\ \mu_2'(2) \\ \mu_3'(2) \\ \mu_4'(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 6 \\ 1 & 14 \end{pmatrix} \begin{pmatrix} \pi_{(1)} \\ \pi_{(2)} \end{pmatrix}.$$

It follows from (3.5) and (3.6) that the corrected moments of X for $k = 2$ are given by

$$\begin{aligned} \mu_2(k=2) &= \pi_{(1)} - \pi_{(1)}^2 + 2\pi_{(2)}, \\ \mu_3(k=2) &= \pi_{(1)} - 3\pi_{(1)}^2 + 2\pi_{(1)}^3 + 6\pi_{(2)} - 6\pi_{(1)}\pi_{(2)}, \\ \mu_4(k=2) &= \pi_{(1)} - 4\pi_{(1)}^2 + 6\pi_{(1)}^3 - 3\pi_{(1)}^4 + 14\pi_{(2)} - 24\pi_{(1)}\pi_{(2)} + 12\pi_{(2)}\pi_{(1)}^2. \end{aligned} \quad (3.7)$$

It follows from (3.3) and (3.4) that the raw moments of X for $k = 3$ are given by

$$\begin{aligned} \mu_1'(k=3) &= \mu = \pi_{(1)}, \\ \mu_2'(k=3) &= \mu + \mu_{(2)}' = \pi_{(1)} + 2\pi_{(2)}, \\ \mu_3'(k=3) &= \pi_{(1)} + 3\left(2!\ \pi_{(2)}\right) + 3!\ \pi_{(3)} = \pi_{(1)} + 6\pi_{(2)} + 6\pi_{(3)}, \\ \mu_4'(k=3) &= \pi_{(1)} + 7\left(2!\ \pi_{(2)}\right) + 6\left(3!\ \pi_{(3)}\right) + 0 = \pi_{(1)} + 14\pi_{(2)} + 36\pi_{(3)}. \end{aligned} \quad (3.8)$$

Which can be presented as

$$\begin{pmatrix} \mu_1'(3) \\ \mu_2'(3) \\ \mu_3'(3) \\ \mu_4'(3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 6 & 6 \\ 1 & 14 & 36 \end{pmatrix} \begin{pmatrix} \pi_{(1)} \\ \pi_{(2)} \\ \pi_{(3)} \end{pmatrix}.$$

It follows from (3.5) and (3.8) that the corrected moments of X for $k = 3$ are given by

$$\begin{aligned} \mu_2(k=3) &= \pi_{(1)} - \pi_{(1)}^2 + 2\pi_{(2)}, \\ \mu_3(k=3) &= \pi_{(1)} - 3\pi_{(1)}^2 + 2\pi_{(1)}^3 + 6\pi_{(2)} + 6\pi_{(3)} - 6\pi_{(1)}\pi_{(2)}, \\ \mu_4(k=3) &= \pi_{(1)} - 4\pi_{(1)}^2 + 6\pi_{(1)}^3 - 3\pi_{(1)}^4 + 14\pi_{(2)} + 36\pi_{(3)} \\ &\quad - 24\pi_{(1)}\pi_{(2)} - 24\pi_{(1)}\pi_{(3)} + 12\pi_{(2)}\pi_{(1)}^2. \end{aligned} \quad (3.9)$$

It follows from (3.3) and (3.4) that the raw moments of X for $k = 4$ are given by

$$\mu_1'(k=4)=\mu=\pi_{(1)},$$
$$\mu_2'(k=4)=\mu+\mu_{(2)}'=\pi_{(1)}+2\pi_{(2)},$$
$$\mu_3'(k=4)=\pi_{(1)}+3\left(2!\ \pi_{(2)}\right)+3!\ \pi_{(3)}=\pi_{(1)}+6\pi_{(2)}+6\pi_{(3)},$$
$$\mu_4'(k=4)=\pi_{(1)}+7\left(2!\ \pi_{(2)}\right)+6\left(3!\ \pi_{(3)}\right)+4!\pi_{(4)}=\pi_{(1)}+14\pi_{(2)}+36\pi_{(3)}+24\pi_{(4)}+0,$$
(3.10)

Which can be presented as

$$\begin{pmatrix}\mu_1'(3)\\ \mu_2'(3)\\ \mu_3'(3)\\ \mu_4'(3)\end{pmatrix}=\begin{pmatrix}1&0&0&0\\1&2&0&0\\1&6&6&0\\1&14&36&24\end{pmatrix}\begin{pmatrix}\pi_{(1)}\\ \pi_{(2)}\\ \pi_{(3)}\\ \pi_{(4)}\end{pmatrix}.$$

It follows from (3.5) and (3.10) that the corrected moments of X for $k=4$ are given by

$$\begin{aligned}\mu_2(k=4)&=\pi_{(1)}-\pi_{(1)}^2+2\pi_{(2)},\\ \mu_3(k=4)&=\pi_{(1)}-3\pi_{(1)}^2+2\pi_{(1)}^3+6\pi_{(2)}+6\pi_{(3)}-6\pi_{(1)}\pi_{(2)},\\ \mu_4(k=4)&=\pi_{(1)}-4\pi_{(1)}^2+6\pi_{(1)}^3-3\pi_{(1)}^4+14\pi_{(2)}+36\pi_{(3)}\\ &\quad+24\pi_{(4)}-24\pi_{(1)}\pi_{(2)}-24\pi_{(1)}\pi_{(3)}+12\pi_{(2)}\pi_{(1)}^2.\end{aligned} \tag{3.11}$$

CONCLUSION

We have tried to prepare a pedagogical account of cardinality between joint and disjoint sets. These problems appear in business mathematics, statistics, and probability theory and are usually solved by drawing a Venn diagram. The presentation in the paper will make learning and teaching the topic amusingly interesting, particularly for students and instructors. Research on other aspects of moments and statistical characterizations of the related probability distribution briefed in Section 3 remains open.

REFERENCES

Björklund, A., Husfeldt, T. and Koivisto, M. (2009). Set Partitioning via Inclusion-Exclusion, *SIAM J. Comput.*, 39(2), 546–563. https://doi.org/10.1137/070683933

Eisen, M. (1969). *Elementary Combinatorial Analysis*. Gordon and Breach Science Publishers. New York.

Hines, W.W. & Montgomery, D.C. (1990). *Probability and Statistics for Engineering and Management Science*. John Wiley, New York.

Johnson, N.L., Kotz, S. and Kemp, A.W. (1993). *Univariate Discrete Distributions*. John Wiley and Sons Inc., New York.

Kolman, B., Anton, H. and Averbach, B. (1992). *Mathematics with Applications for the Management, Life, and Social Sciences*. Saunders College Publishing, New York.

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