A Note on the Tower of Hanoi Problem with Evildoers

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ABSTRACT

A recent literature considers the variant of the classical Tower of Hanoi problem with n (≥ 1) discs, where r ($1 \leq r < n$) discs are evildoers, each of which can be placed directly on top of a smaller disc any number of times. Letting $E(n, r)$ be the minimum number of moves required to solve the new variant, an explicit form of $E(n, r)$ is available which depends on a positive integer constant $\boldsymbol{N}$. This study investigates the properties of $\boldsymbol{N}$.

Key words:
Divine rule, evildoer, Hanoi tower problem

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INTRODUCTION

The Tower of Hanoi problem, in its general form, is as follows: Given are n (≥ 1) discs d_1, d_2, …, d_n of different sizes, and three pegs, S, P and D. At the beginning of the game, the discs rest on the *source peg*, S, in a *tower* in increasing order, from top to bottom (so that, the topmost disc is d_1 and the bottommost disc is d_n). The objective is to transfer the tower to the *destination peg*, D, in a minimum number of moves, where each move can shift only the topmost disc from one peg to another, under the "divine rule" that no disc can ever be placed on a smaller disc. The initial configuration is shown schematically by Figure 1 below.

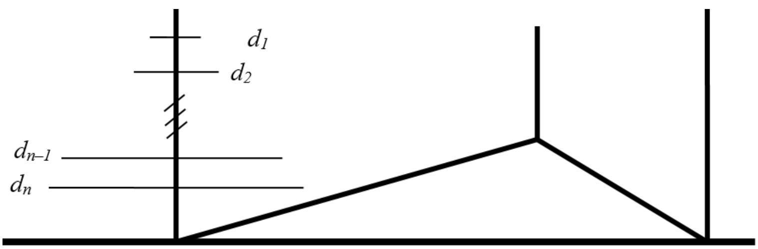

Figure 1: Initial state in the Tower of Hanoi problem

Over the last decades, the Tower of Hanoi problem has seen many variations and generalizations, some of which have been reviewed by Wu and Chen (1993), Majumdar (2012), Majumdar (2013), Majumdar (2018), Hinz et al. (2018) and Majumdar and Islam (2019). Chen et al. (2007) have introduced a new variant of the Tower of Hanoi problem, which allows r $(1 \le r < n)$ violations of the "divine rule". In the new variant, the problem is to shift the tower of n (≥ 1) discs from the peg S to the peg D in a minimum number of moves, where for (at most) r moves, some disc may be placed directly on top of a smaller one.

Let the minimum number of moves required to solve the above problem be $S(n, r)$. Then, an explicit form of $S(n, r)$, due to Chen et al. (2007) (in a slightly modified form), is given below.

Theorem 1.1: For any $n \ge 1$, $r \ge 1$,

$$S(n,r) = \begin{cases} 2n-1, & if \;\; 1 \le n \le r+2 \\ 4n-2r-5, & if \;\; r+2 \le n \le 2r+3 \\ 2^{n-2r}+6r-1, & if \;\; n \ge 2r+3 \end{cases}$$

Chen et al. (2007) posed the following problem: In the classical Tower of Hanoi problem, any r (of the n (≥ 1)) discs are evildoers, where each *evildoer* can be placed (directly) on top of a smaller disc any number of times. The objective is to find the minimum number of moves required to solve the new problem. Some background materials are discussed in Section 2. Main results are made in Section 3. Finally, Section 4 includes concluding remarks.

MATERIALS AND METHODS

The problem posed by Chen et al. (2007) was taken up by Majumdar (2019) and Majumdar & Islam (2020). Let $E(n, r)$ be the minimum number of moves required to solve the above variant. Then, an explicit form of $E(n, r)$, due to Majumdar (2019), is given as follows:

Theorem 2.1: Let, for some integers m and j,

$$n = (r+1)m + j;\; m \ge 1,\; 1 \le j \le r.$$

Then, there exists an integer $\boldsymbol{N}$ such that for $n \ge \boldsymbol{N}$,

$$E(n, r) = 2^{n-2r-1} + (r+j+1)2^{m-1} + 10r - 1$$

(with $E(n, r) = S(n, r)$ if $1 \le n < \boldsymbol{N}$).

It may be recalled that, to find $E(n, r)$, the scheme below is followed.

Step 1: Move the tower of the topmost $n-2r-2$ discs, $d_1, d_2, \ldots, d_{n-2r-2}$, from the peg S to the peg D.

Step 2: With the next $2r$ discs on the peg S, form r pairs. For each pair of discs (d_i, d_{i+1}), d_i is first moved to the peg P, next d_{i+1} is shifted to the peg D (violating the "divine rule"), and then the disc d_i is moved to the peg D. This step requires $3r$ moves, and the "divine rule" is violated r times.

Step 3: Move the disc d_{n-1} (from the peg S) to the peg P.

Step 4: Transfer the topmost $2r$ discs on the destination peg D, one-by-one, the r discs d_{n-3}, $d_{n-5}, \ldots, d_{n-2r-1}$ (in this order) to the peg P, and the r evildoers $d_{n-2}, d_{n-4}, \ldots, d_{n-2r}$ (in this order) to the peg S. This step requires $2r$ moves.

After Step 4, we have the tower of discs $d_1, d_2, \ldots, d_{n-2r-2}$ on the peg D, which is now divided into $r+1$ subtowers, $T_1, T_2, \ldots, T_{r+1}$, of sizes $m_1, m_2, \ldots, m_{r+1}$ respectively, where $m_1 + m_2 + \ldots + m_{r+1} = n - 2r - 2$, and $m_1, m_2, \ldots, m_{r+1}$ are determined so as to minimize $2^{m_1} + 2^{m_2} + \ldots + 2^{m_r} + 2^{m_{r+1}}$.

Step 5: Move the $r+1$ subtowers $T_1, T_2, \ldots, T_{r+1}$, using the r evildoers (on the peg S), one-by-one, to the peg P. This is done as follows: Move the tower T_k, followed by the transfer of the evildoer disc $d_{n-2r+2(k-1)}$, for $1 \le k \le r$, and then transfer the largest tower T_{r+1} on top of the evildoer disc d_{n-2}.

Step 6: Move the largest disc d_n (from the peg S) to the peg D.

Step 7: Move the subtowers $T_{r+1}, T_r, \ldots, T_1$ (in this order, on the peg P) to the peg S and the evildoers to the peg D.

After Step 7, we have the tower of smallest $n-2r-2$ discs on the peg S, and the r evildoer discs $d_{n-2r}, d_{n-2r+2}, \ldots, d_{n-2}$ (in this order) on the peg D on top of the disc d_n.

Step 8: Place the evildoer d_{n-2r} (from the peg D) to the peg S on top of the tower of the $n-2r-2$ discs, followed by the transfer of the disc d_{n-2r-1} (from the peg P) on top of d_{n-2r}. We continue this process, and finally, after moving the evildoer disc d_{n-2} (from the peg D) to the peg S, we transfer the disc d_{n-3} (from the peg P) on top of the disc d_{n-2}.

After Step 8, we have the pair of discs (d_i, d_{i+1}), $i = n-3, n-5, \ldots, n-2r-1$ (in this order) on the peg S, and the free disc d_{n-1} on the peg P.

Step 9: Move the disc d_{n-1} to the peg D.

Step 10: With the pair of discs (d_i, d_{i+1}) on the peg S, the disc d_i is first moved to the peg P, then the disc d_{i+1} is transferred to the peg D, and finally, the disc d_i is moved to the peg D.

Step 11: Move the tower of $n-2r-2$ discs on the peg S to the peg D.

In course of proving Theorem 2.1, it has also been proved that, for $n \ge \mathbf{N}$,

$$E(n,r) < S(n,r)$$

if and only if

$$2^{m-2}[2^{(m-2)r+j} - (r+j+1)] > 2r. \tag{1}$$

Then, for fixed r (≥ 1), there are integers m and j ($0 \le j \le r$) such that (1) holds. With the minimum such m and j,

$$\mathbf{N} = (r+1)m + j. \tag{2}$$

This paper studies some of the properties of $\mathbf{N}$. This is done in Section 3.

RESULTS AND DISCUSSION

In this section, we consider in detail some characteristics of the integer $\mathbf{N}$, given in (2). To do so, let

$$F(m,r,j) \equiv 2^{(m-2)r+j} - (r+j+1);\ m \ge 2,\ r \ge 1,\ 0 \le j \le r. \tag{3}$$

Then, we have the following result, giving some monotonic properties of $F(m,r,j)$.

Lemma 3.1: The following results hold:

(a) for r and j fixed, $F(m,r,j)$ is strictly increasing in $m \geq 2$,

(b) for m (≥ 2) and r fixed, $F(m,r,j)$ is increasing in $j \geq 0$,

(c) for j fixed, $F(2,r,j)$ is strictly decreasing in r, and for m (≥ 3) and j fixed, $F(m,r,j)$ is strictly increasing in r,

(d) for j fixed, $F(m+1,r,j) > F(m,r+1,j)$ if $r > m-2$,

(e) for m fixed, $F(m,r+1,j) > F(m,r,j+1)$ if and only if $m \geq 3$,

(f) for r fixed, $F(m+1,r,j) > F(m,r,j+1)$.

Proof: Part (a) is immediate from the defining equation (3).

Now, for $m \geq 2$ and $r \geq 1$ fixed,

$$F(m,r,j+1) \equiv 2^{(m-2)r+j+1} - (r+j+2) \geq 2^{(m-2)r+j} - (r+j+1) \equiv F(m,r,j)$$

if and only if

$$2^{(m-2)r+j} \geq 1,$$

which is true for all $j \geq 0$ (for all $m \geq 2$). This proves part (b).

Again, for $m \geq 2$ and j $(0 \leq j \leq r)$ fixed,

$$\begin{aligned} F(m,r+1,j) - F(m,r,j) &= [2^{(m-2)(r+1)+j} - (r+j+2)] - [2^{(m-2)r+j} - (r+j+1)] \\ &= 2^{(m-2)r+j}(2^{m-2} - 1) - 1, \end{aligned}$$

which shows that, for any r with $r+j \geq 2$,

$$F(m,r+1,j) - F(m,r,j) > 2 \text{ if } m \geq 3.$$

This establishes part (c) of the lemma.

Now,

$$F(m+1,r,j) \equiv 2^{(m-1)(r+1)+j} - (r+j+2) > 2^{(m-2)(r+1)+j} - (r+j+2) \equiv F(m,r+1,j)$$

if and only if

$$2^{(m-2)r+j}(2^r - 2^{m-2}) > -1,$$

which holds true if $r > m-2$. Thus, part (d) is proved.

Again,

$$F(m,r+1,j) \equiv 2^{(m-2)(r+1)+j} - (r+j+2) > 2^{(m-2)r+j+1} - (r+j+2) \equiv F(m,r,j+1)$$

if and only if

$$2^{m-2} > 2,$$

from which part (e) follows readily.

Finally, since

$$F(m+1,r,j) \equiv 2^{(m-1)r+j} - (r+j+1) > 2^{(m-2)r+j+1} - (r+j+2) \equiv F(m,r,j+1)$$

if and only if

$$2^{(m-2)r+j}(2^r - 1) > -1,$$

part (f) of the lemma follows.

The proof of Lemma 3.1 shows that, for m (≥ 2) and r (≥ 1) fixed, $F(m,r,j)$ is strictly increasing in $j \geq 1$. One consequence of Lemma 3.1 is the following, whose proof is simple, and is omitted here.

Corollary 3.1: Let $2^{m-2}F(m,r,j) > 2r$ for some integers m (≥ 2), r (≥ 1) and j $(0 \leq j \leq r)$. Then,

$$2^{m-2}F(m+1,r,j) > 2r,$$

$$2^{m-2}F(m,r,j+1) > 2r \text{ (where } 0 \leq j \leq r\text{)},$$

$$2^{m-2}F(m,r+1,j) > 2(r+1) \text{ if } r \geq 2 \text{ and } m \geq 3.$$

Lemma 3.2: For any $r \geq 1$, $F(2,r+1,j+1) > F(2,r,j)$ if and only if $2 \leq j \leq r$.

Proof: Since

$$F(2,r+1,j+1) - F(2,r,j) = [2^{j+1} - (r+j+3)] - [2^j - (r+j+1)] = 2^j - 2,$$

the result follows.

Corollary 3.2: If $F(2,r,j) > 2r$ for some integers r and j (with $3 \leq j \leq r$), then

$$F(2,r+1,j+1) > 2(r+1).$$

Proof: follows from Lemma 2.2.

Theorem 2.1 gives the minimum number of moves required to solve the Tower of Hanoi problem with n (≥ 1) discs and r (≥ 1) evildoers. The minimum number of moves, denoted by $E(n,r)$, depends on the integer $\boldsymbol{N}(r)$, where $\boldsymbol{N}(r)$ is defined through the equation (2). Given r, the problem is to find the minimum m and j $(0 \leq j \leq r)$ such that the inequality (1) is satisfied.

When $r = 1$, the condition (1) becomes

$$2^{m-2}(2^{m-2+j} - 2 - j) > 2; \; j \in \{0, 1\}. \tag{4}$$

It can easily be verified that the inequality (4) is not satisfied when $m = 1, 2, 3$. When $m = 4$, the inequality (4) reads as

$$4(2^{2+j} - 2 - j) > 2; \; j \in \{0, 1\},$$

which is satisfied when $j = 0$. Therefore, from (2), the corresponding $\boldsymbol{N}$ is given by $\boldsymbol{N}(1) = 8$. Thus,

$$E(n,1) < S(n,1) \text{ for all } n \geq 8.$$

With $r = 2$, the inequality (1) takes the form

$$2^{m-2}[2^{2(m-2)+j} - 3 - j] > 2, \tag{5}$$

which is not satisfied when $m = 2$. However, the inequality (5) is satisfied with $m = 3$ and $j = 1$, so that the corresponding $\boldsymbol{N}(2) = 10$, and hence,

$E(n,2) < S(n,2)$ for all $n \geq 10$.

Thus, when $r = 2$, the minimum values of m and j $(0 \leq j \leq r)$ satisfying the inequality (1) are $m = 3$, $j = 1$, so that $2F(3,2,1) > 4$. Then, by part (c) of Corollary 2.1,

$2F(3,3,1) > 6$.

Thus, when $r = 3$, in order that the condition (1) is satisfied, we must have $m \leq 3$. In fact, when $r = 3$, the inequality (1) reads as

$$2^{m-2}[2^{3(m-2)+j} - (j+4)] > 6,$$

which is satisfied with $m = 3$ and $j = 0$. Therefore, the corresponding $\boldsymbol{N}(3) = 12$, and hence,

$E(n,3) < S(n,3)$ for all $n \geq 12$.

Similarly, when $r = 4$, in condition (1), we must have $m \leq 3$. In fact, in this case, (1) is satisfied with m=3 and $j = 0$, so that the corresponding $\boldsymbol{N}(4) = 15$. However, when $r = 5$, the condition (1) is satisfied with $m = 2$ and $j = 5$. Note that, when $m = 2$, the condition (1) becomes simply

$$2^j - (r + j + 1) > 2r. \tag{6}$$

When $r = 5$, the condition (6) is satisfied with $j = 5$, so that the corresponding $\boldsymbol{N}(5) = 17$.

Then, by Corollary 3.2, when $r = 6$, the condition (6) is satisfied with $j \leq 6$. It can easily be verified that, in such a case, the condition (6) is satisfied with $j = 5$, so that the corresponding $\boldsymbol{N}(6) = 17$.

We now state and prove the following results.

Theorem 3.1: For $r \geq 5$, the condition (1) is satisfied with (minimum) $m = 2$.

Proof: The theorem is clearly true for $r = 5$ with $j = 5$. Then, by Corollary 3.2, the result is true for r =6 with $j \leq 6$. Continuing the argument, the result follows.

Theorem 3.2: $\boldsymbol{N} = \boldsymbol{N}(r)$ is strictly increasing in r.

Proof: By Theorem 3.1, for $r \geq 5$, the condition (1) is satisfied with $m = 2$. Let for such an r, the minimum j satisfying the condition (6) be J, so that

$$2^J - (r + J + 1) > 2r,\ 2^{J-1} - (r + J) < 2r. \tag{7}$$

Recall that, for $r = 5$, the conditions in (7) are satisfied with $J = 5$. Now, consider the problem of finding the minimum j satisfying the condition

$$2^j - (r + j + 2) > 2(r + 1). \tag{8}$$

We now prove that $j \neq J - 1$, for otherwise, from the inequality (8), we have

$$2^{J-1} - (r + J + 1) > 2(r + 1),$$

which violates the right hand side inequality in (7). Then, by (2)

$$\boldsymbol{N}(r+1) = 2(r+2) + j \geq 2(r+2) + J > 2(r+1) + \mathrm{J} = \boldsymbol{N}(r).$$

Thus, $\boldsymbol{N}(r)$ is strictly increasing in $r \geq 5$. The values of $\boldsymbol{N}(r)$ for $r < 5$ now establish the theorem.

Conclusions

This paper gives some properties satisfied by the integer $\boldsymbol{N} = \boldsymbol{N}(r)$. One important property is that, for $r \geq 5$, $m = 2$; moreover, in such a case, if for any r, the inequality (1) is satisfied for $j = J$, then (1) is satisfied for $r+1$ with j such that $J \leq j \leq J+1$. This may enable us to calculate $\boldsymbol{N} = \boldsymbol{N}(r)$ recursively in r.

Table 1: Values of N for small values of r

r	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{N}$	8	10	12	15	17	19	21	23	26	28

Table 1 above gives the values of $\boldsymbol{N} = \boldsymbol{N}(r)$ for small values of r. The results derived in the paper may open a new research direction.

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