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Improved Understanding of k-ε Turbulence Model for Non-homogeneous Two-Phase Flows in Industrial Combustor

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ABSTRACT

This paper presents development of k-ε turbulence model which can be applied for non-homogeneous two-phase turbulent flows. The improvement of governing equations of k-ε turbulence model is based on the decomposition of the two-phase flows into the gas phase and the second phase. Thus, the turbulent kinetic energy k and the turbulent dissipation rate ε variables in k-ε two-equation model are substituted by the turbulent kinetic energy kg of the gas phase, the turbulent kinetic energy kp of the second phase, the turbulent dissipation rate εg of the gas phase and the turbulent dissipation rate εp of the second phase. The new turbulence model is kg-kp-εg-εp four-equation model which can be used to solve all of two-phase flows in nature studies and engineering applications.

Keywords: Turbulence model, Non-homogeneous,Two-phase flows, Turbulent kinetic energy, Turbulent dissipation rate;

1. INTRODUCTION

History effects are taken into account by the one-equation and two-equation models, where the convection and diffusion of turbulence is modeled by transport equations. The most widely used one-equation turbulence model is due to Spalart and Allmaras (1992), which is based on an eddy-viscosity like variable. The model is numerical very stable and easy to implement on structured as well as unstructured grids. In the case of the two-equation models, practically all approaches employ the transport equation for the turbulent kinetic energy. Among a large number of two-equation models, the k-ε model of Launder and Spalding and the k-ω model of Wilcox (1988) are most often used in engineering applications. They offer a reasonable compromise between computational effort and accuracy.

The k-ε turbulence model is the most widely employed two-equation eddy viscosity model. It is based on the solution of equations for the turbulent kinetic energy and the turbulent dissipation rate. The historic roots of the k-ε model reach to the work of Chou (Chou 1945). During the 1970's, various formulations of the model were proposed. The most important contributions were due to Jones and Launder, Launder and Sharma (Launder 1974) as well as due to Launder and Spalding (Launder 1974).

The first approach for the approximate treatment of turbulent flows was presented by Reynolds in 1895. The methodology is based on the decomposition of the flow variables into a mean and a fluctuating part Eqn. (1). The governing equations are solved for the mean values, which are the most interesting for engineering applications. The velocity components is substituted by (Blazek 2001).

$$v_i = \bar{v}_i + v_i^{'} \tag{1}$$

where the mean value is denoted by an overbar and the turbulent fluctuation by a prime (Figure 1).
Appropriate for general turbulence

$$\bar{v}_i = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} v_i \tag{2}$$

Hence, the mean value $\bar{v}_i$ still remains a function of time and space coordinates, the average of the fluctuating part is zero, i.e., $\bar{v}_i^{'} = 0$. However, it can be easily seen that $\overline{v_i^{'} v_i^{'}} \neq 0$ and $\overline{v_i^{'} v_j^{'}} \neq 0$ if both turbulent velocity components are correlated.

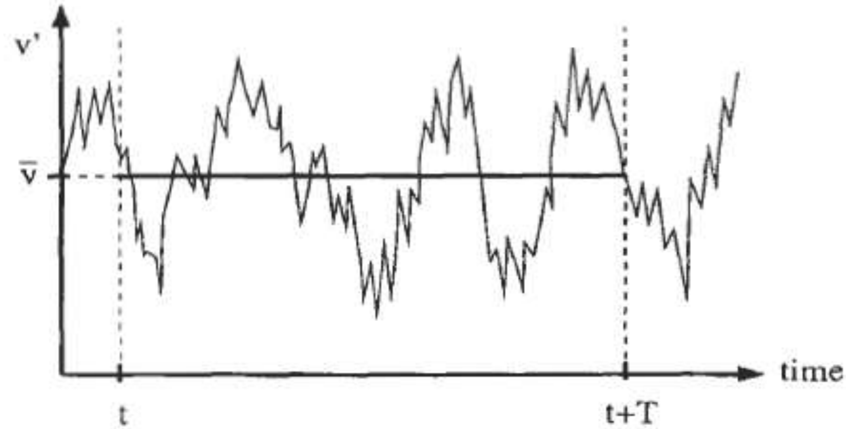

Figure 1. Reynolds averaging illustration of turbulent velocity ($\bar{v}, v'$)

The sum of the normal stresses divided by density defines the turbulent kinetic energy, i.e.,

$$k = \frac{1}{2}\overline{v_i^{'} v_i^{'}} = \frac{1}{2}\left[\overline{\left(v_1^{'}\right)^2} + \overline{\left(v_2^{'}\right)^2} + \overline{\left(v_3^{'}\right)^2}\right] \tag{3}$$

The differential form of a low Reynolds number k-ε model can be written as

$$\frac{\partial(\rho k)}{\partial t} + \frac{\partial(\rho k v_j)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\mu_L + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + \tau_{ij}^F S_{ij} - \rho\varepsilon \tag{4}$$

$$\frac{\partial(\rho\varepsilon)}{\partial t} + \frac{\partial(\rho v_j \varepsilon)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\mu_L + \frac{\mu_T}{\sigma_k}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + C_{\varepsilon 1} f_{\varepsilon 1}\frac{\varepsilon}{k}\tau_{ij}^F S_{ij} - C_{\varepsilon 2} f_{\varepsilon 2}\rho\frac{\varepsilon^2}{k} + \phi_\varepsilon \tag{5}$$

In case where the turbulent flow is non-homogeneous two-phase, the Eqn. (3) is not suitable application unless we assume that the flow is homogeneous flow or one phase flow, because the velocities of non-homogeneous two-phase flow include velocity of the gas phase and velocity of the second phase. So that, Eqns. (3)÷(5) must be improved for non-homogeneous two-phase flows.

2. GOVERNING EQUATIONS AND SOLUTION METHOD

2.1Governing equations

The flow-system equations of two-phase turbulent flows were established by Schraiber et al. as follows:

$$\begin{cases} \dfrac{\partial \rho_g}{\partial t} + \sum_j \left(\dfrac{\partial \rho_{gj} v_{gj}}{\partial x_j} \right) = 0 \\ \dfrac{\partial \rho_p}{\partial t} + \sum_j \left(\dfrac{\partial \rho_{pj} v_{pj}}{\partial x_j} \right) = 0 \\ \dfrac{\partial v_{gi}}{\partial t} + \sum_j v_{gj} \dfrac{\partial v_{gi}}{\partial x_j} = -\dfrac{1}{\rho_g} \dfrac{\partial p}{\partial x_i} + \dfrac{1}{\rho_g} \sum_j \dfrac{\partial \tau_{gij}}{\partial x_j} - \dfrac{1}{\rho_g} F_i \\ \dfrac{\partial v_{pi}}{\partial t} + \sum_j v_{pj} \dfrac{\partial v_{pi}}{\partial x_j} = \dfrac{1}{\rho_p} \sum_j \dfrac{\partial \tau_{pij}}{\partial x_j} + \dfrac{1}{\rho_p} F_i \end{cases} \tag{6}$$

Following Erlebacher et al. (1992), the flow variables are decomposed into mean and turbulent fluctuationvelocities as Eqns. (7)

$$\begin{aligned} v_g &= \bar{v}_g + v_g^{'} \\ v_p &= \bar{v}_p + v_p^{'} \end{aligned} \tag{7}$$

where g, $p$are the symbols for the gas phase and the second phase; $i, j = 1, 2, 3$ (according to the coordinates); v_{gi}, v_{gj}, v_{pi}, v_{pj} are the velocity components; ρ_g,ρ_p are the density of the phases; p is the pressure; τ_{gij}, τ_{pij} are the stress components of the phases; F_i is the interaction force between the phases; and the prime symbol means a turbulent component.

Because the average components always satisfy Eqns. (6), the above system equations for the turbulent components in two-dimension coordinates becomes

$$\frac{\partial \rho_g}{\partial t} + \frac{\partial \left(\rho_g u_g^{'} \right)}{\partial x} + \frac{\partial \left(\rho_g v_g^{'} \right)}{\partial y} = 0 \tag{8}$$

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial \left(\rho_p u_p^{'} \right)}{\partial x} + \frac{\partial \left(\rho_p v_p^{'} \right)}{\partial y} = 0 \tag{9}$$

$$\begin{aligned} &\frac{\partial u_g^{'}}{\partial t} + \left(u_g^{'} \frac{\partial \bar{u}_g}{\partial x} + v_g^{'} \frac{\partial \bar{u}_g}{\partial y} \right) + \left(\bar{u}_g \frac{\partial u_g^{'}}{\partial x} + \bar{v}_g \frac{\partial u_g^{'}}{\partial y} \right) + \left(u_g^{'} \frac{\partial u_g^{'}}{\partial x} + v_g^{'} \frac{\partial u_g^{'}}{\partial y} \right) \\ &= -\frac{1}{\rho_g} \frac{\partial p^{'}}{\partial x} + \frac{1}{\rho_g} \left(\frac{\partial \tau_{gxx}^{'}}{\partial x} + \frac{\partial \tau_{gxy}^{'}}{\partial y} \right) - \frac{1}{\rho_g} F_x^{'} \end{aligned}$$

(10)

$$\frac{\partial v_g'}{\partial t}+\left(u_g'\frac{\partial \bar{v}_g}{\partial x}+v_g'\frac{\partial \bar{v}_g}{\partial y}\right)+\left(\bar{u}_g\frac{\partial v_g'}{\partial x}+\bar{v}_g\frac{\partial v_g'}{\partial y}\right)+\left(u_g'\frac{\partial v_g'}{\partial x}+v_g'\frac{\partial v_g'}{\partial y}\right)$$
$$=-\frac{1}{\rho_g}\frac{\partial p'}{\partial x}+\frac{1}{\rho_g}\left(\frac{\partial \tau_{gxy}'}{\partial x}+\frac{\partial \tau_{gyy}'}{\partial y}\right)-\frac{1}{\rho_g}F_y' \qquad (11)$$

$$\frac{\partial u_p'}{\partial t}+\left(u_p'\frac{\partial \bar{u}_p}{\partial x}+v_p'\frac{\partial \bar{u}_p}{\partial y}\right)+\left(\bar{u}_p\frac{\partial u_p'}{\partial x}+\bar{v}_p\frac{\partial u_p'}{\partial y}\right)+\left(u_p'\frac{\partial u_p'}{\partial x}+v_p'\frac{\partial u_p'}{\partial y}\right)$$
$$=\frac{1}{\rho_p}\left(\frac{\partial \tau_{pxx}'}{\partial x}+\frac{\partial \tau_{pxy}'}{\partial y}\right)+\frac{1}{\rho_p}F_x' \qquad (12)$$

$$\frac{\partial v_p'}{\partial t}+\left(u_p'\frac{\partial \bar{v}_p}{\partial x}+v_p'\frac{\partial \bar{v}_p}{\partial y}\right)+\left(\bar{u}_p\frac{\partial v_p'}{\partial x}+\bar{v}_p\frac{\partial v_p'}{\partial y}\right)+\left(u_p'\frac{\partial v_p'}{\partial x}+v_p'\frac{\partial v_p'}{\partial y}\right)$$
$$=\frac{1}{\rho_p}\left(\frac{\partial \tau_{pxy}'}{\partial x}+\frac{\partial \tau_{pyy}'}{\partial y}\right)+\frac{1}{\rho_p}F_y' \qquad (13)$$

Using Eqns. (8)÷(13) for the turbulent components, we can set up equations for the turbulent kinetic energy (k_g–k_p) and the turbulent dissipation rate (ε_g–ε_p). Because the equations k_g–ε_g of the gas phase have already been developed by other authors (Launder 1974; Blazek 2001), so in this paper we just have to write equations for the turbulent kinetic energy $k_p$and turbulent dissipation rate ε_p of the second phase.

$$\rho_g\frac{\partial k_g}{\partial t}+\rho_g u_g\frac{\partial k_g}{\partial x}+\rho_g v_g\frac{\partial k_g}{\partial y}=\frac{\partial}{\partial y}\left[\frac{\mu_g}{\sigma_k}\left(\frac{\partial k_g}{\partial y}\right)\right]+\mu_g\left(\frac{\partial u_g}{\partial y}\right)^2-\rho_g\varepsilon_g-\varepsilon^* \quad (14)$$

$$\rho_g\frac{\partial \varepsilon_g}{\partial t}+\rho_g u_g\frac{\partial \varepsilon_g}{\partial x}+\rho_g v_g\frac{\partial \varepsilon_g}{\partial y}=\frac{\partial}{\partial y}\left[\frac{\mu_g}{\sigma_\varepsilon}\left(\frac{\partial \varepsilon_g}{\partial y}\right)\right]+C_{\varepsilon 1}\mu_g\frac{\varepsilon_g}{k_g}\left(\frac{\partial v_g}{\partial y}\right)^2-C_{\varepsilon 2}\frac{\varepsilon_g^2}{k_g}-\phi_\varepsilon \quad (15)$$

2.2 Turbulent kinetic energy equation of the second phase k_p

Multiplying Eqn. (12) by the turbulent fluctuation of velocity u'_p, and Eqn. (13) by the turbulent fluctuation of velocity v'_p, we get

$$u_p'\frac{\partial u_p'}{\partial t}+\left(u_p'u_p'\frac{\partial \bar{u}_p}{\partial x}+u_p'v_p'\frac{\partial \bar{u}_p}{\partial y}\right)+\left(u_p'\bar{u}_p\frac{\partial u_p'}{\partial x}+u_p'\bar{v}_p\frac{\partial u_p'}{\partial y}\right)+\left(u_p'u_p'\frac{\partial u_p'}{\partial x}+u_p'v_p'\frac{\partial u_p'}{\partial y}\right) \quad (14)$$
$$=\frac{1}{\rho_p}\left(u_p'\frac{\partial \tau_{pxx}'}{\partial x}+u_p'\frac{\partial \tau_{pxy}'}{\partial y}\right)+\frac{1}{\rho_p}u_p'F_x'$$

$$v_p' \frac{\partial v_p'}{\partial t} + \left(v_p' u_p' \frac{\partial \bar{v}_p}{\partial x} + v_p' v_p' \frac{\partial \bar{v}_p}{\partial y} \right) + \left(v_p' \bar{u}_p \frac{\partial v_p'}{\partial x} + v_p' \bar{v}_p \frac{\partial v_p'}{\partial y} \right) + \left(v_p' u_p' \frac{\partial v_p'}{\partial x} + v_p' v_p' \frac{\partial v_p'}{\partial y} \right)$$
$$= \frac{1}{\rho_p} \left(v_p' \frac{\partial \tau_{pxy}'}{\partial x} + v_p' \frac{\partial \tau_{pyy}'}{\partial y} \right) + \frac{1}{\rho_p} v_p' F_y'$$

(15)

Adding both sides of Eqn. (14) and Eqn. (15) we get the Eqn. (16)

$$\rho_p \left(u_p' \frac{\partial u_p'}{\partial t} + v_p' \frac{\partial v_p'}{\partial t} \right)$$
$$+ \rho_p \left(u_p' u_p' \frac{\partial \bar{u}_p}{\partial x} + u_p' \bar{u}_p \frac{\partial u_p'}{\partial x} + u_p' u_p' \frac{\partial u_p'}{\partial x} + v_p' u_p' \frac{\partial \bar{v}_p}{\partial x} + v_p' \bar{u}_p \frac{\partial v_p'}{\partial x} + v_p' u_p' \frac{\partial v_p'}{\partial x} \right) \quad (16)$$
$$+ \rho_p \left(u_p' v_p' \frac{\partial \bar{u}_p}{\partial y} + u_p' \bar{v}_p \frac{\partial u_p'}{\partial y} + u_p' v_p' \frac{\partial u_p'}{\partial y} + v_p' v_p' \frac{\partial \bar{v}_p}{\partial y} + v_p' \bar{v}_p \frac{\partial v_p'}{\partial y} + v_p' v_p' \frac{\partial v_p'}{\partial y} \right)$$
$$= \left[\left(u_p' \frac{\partial \tau_{pxx}'}{\partial x} + u_p' \frac{\partial \tau_{pxy}'}{\partial y} \right) + \left(v_p' \frac{\partial \tau_{pxy}'}{\partial x} + v_p' \frac{\partial \tau_{pyy}'}{\partial y} \right) \right] + \left(u_p' F_x' + v_p' F_y' \right)$$

The turbulent kinetic energy k_p in this equation is defined as Eqn. (17)

$$k_p = \frac{1}{2} \left(u_p'^2 + v_p'^2 \right) \quad (17)$$

So that, the first expression in the left-hand side of Eqn. (16) can be developed

$$\rho_p \left(u_p' \frac{\partial u_p'}{\partial t} + v_p' \frac{\partial v_p'}{\partial t} \right) = \rho_p \frac{\partial}{\partial t} \left[\frac{1}{2} \left(u_p'^2 + v_p'^2 \right) \right] = \rho_p \frac{\partial k_p}{\partial t} \quad (18)$$

The second expression in the left-hand side of Eqn. (16) can be developed

$$\rho_p \left(u_p' u_p' \frac{\partial \bar{u}_p}{\partial x} + u_p' \bar{u}_p \frac{\partial u_p'}{\partial x} + u_p' u_p' \frac{\partial u_p'}{\partial x} + v_p' u_p' \frac{\partial \bar{v}_p}{\partial x} + v_p' \bar{u}_p \frac{\partial v_p'}{\partial x} + v_p' u_p' \frac{\partial v_p'}{\partial x} \right) \quad (19)$$
$$= \rho_p \left[u_p' \left(u_p' \frac{\partial \bar{u}_p}{\partial x} + v_p' \frac{\partial \bar{v}_p}{\partial x} \right) \right] + \rho_p \left[\frac{\left(u_p' + \bar{u}_p \right)}{2} \frac{\partial \left(u_p'^2 + v_p'^2 \right)}{\partial x} \right] = \rho_p u_p \frac{\partial k_p}{\partial x}$$

The third expression in the left-hand side of Eqn. (16) can be developed

$$\rho_p \left(u_p' v_p' \frac{\partial \bar{u}_p}{\partial y} + u_p' \bar{v}_p \frac{\partial u_p'}{\partial y} + u_p' v_p' \frac{\partial u_p'}{\partial y} + v_p' v_p' \frac{\partial \bar{v}_p}{\partial y} + v_p' \bar{v}_p \frac{\partial v_p'}{\partial y} + v_p' v_p' \frac{\partial v_p'}{\partial y} \right) \quad (20)$$
$$= \rho_p \left[v_p' \left(u_p' \frac{\partial \bar{u}_p}{\partial y} + v_p' \frac{\partial \bar{v}_p}{\partial y} \right) \right] + \rho_p \left[\frac{v_p}{2} \frac{\partial \left(u_p'^2 + v_p'^2 \right)}{\partial y} \right] = \rho_p v_p \frac{\partial k_p}{\partial y}$$

The first expression in the right-hand side of Eqn. (16) can be developed

$$\left[\left(u'_p\frac{\partial\tau'_{pxx}}{\partial x}+u'_p\frac{\partial\tau'_{pxy}}{\partial y}\right)+\left(v'_p\frac{\partial\tau'_{pxy}}{\partial x}+v'_p\frac{\partial\tau'_{pyy}}{\partial y}\right)\right]$$
$$=\left[\left(u'_p\frac{\partial\tau'_{pxx}}{\partial x}+v'_p\frac{\partial\tau'_{pxy}}{\partial x}\right)+\left(u'_p\frac{\partial\tau'_{pxy}}{\partial y}+v'_p\frac{\partial\tau'_{pyy}}{\partial y}\right)\right] \quad (21)$$
$$=\left[\frac{\partial}{\partial x}\left(u'_p\tau'_{pxx}\right)+\frac{\partial}{\partial x}\left(v'_p\tau'_{pxy}\right)+\frac{\partial}{\partial y}\left(u'_p\tau'_{pxy}\right)+\frac{\partial}{\partial y}\left(v'_p\tau'_{pyy}\right)\right]$$
$$-\left[\frac{\partial u'_p}{\partial x}\tau'_{pxx}+\frac{\partial v'_p}{\partial x}\tau'_{pxy}+\frac{\partial u'_p}{\partial y}\tau'_{pxy}+\frac{\partial v'_p}{\partial y}\tau'_{pyy}\right]$$

The components of the viscous stress tensor are defined by the relations [8, 9]

$$\tau'_{pxy}=\mu_p\left(\frac{\partial v'_p}{\partial x}+\frac{\partial u'_p}{\partial y}\right)$$
$$\tau'_{pxx}=\lambda.\nabla U+2\mu_p\frac{\partial u'_p}{\partial x} \quad (22)$$
$$\tau'_{pyy}=\lambda.\nabla U+2\mu_p\frac{\partial v'_p}{\partial y}$$

Substituting Eqns. (22) into Eqn. (21) and rearranging, we get the following equation:

$$\left[\left(u'_p\frac{\partial\tau'_{pxx}}{\partial x}+u'_p\frac{\partial\tau'_{pxy}}{\partial y}\right)+\left(v'_p\frac{\partial\tau'_{pxy}}{\partial x}+v'_p\frac{\partial\tau'_{pyy}}{\partial y}\right)\right]$$
$$=\mu_p\left[2\frac{\partial}{\partial x}\left(u'_p\frac{\partial u'_p}{\partial x}\right)+2\frac{\partial}{\partial y}\left(v'_p\frac{\partial v'_p}{\partial y}\right)+\frac{\partial}{\partial x}\left(v'_p\frac{\partial v'_p}{\partial x}+v'_p\frac{\partial u'_p}{\partial y}\right)\right. \quad (23)$$
$$\left.+\frac{\partial}{\partial y}\left(u'_p\frac{\partial v'_p}{\partial x}+u'_p\frac{\partial u'_p}{\partial y}\right)\right]-\mu_p\left(3\frac{\partial u'_p}{\partial x}\frac{\partial v'_p}{\partial x}+\frac{\partial u'_p}{\partial x}\frac{\partial u'_p}{\partial y}+\frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial y}+3\frac{\partial u'_p}{\partial y}\frac{\partial v'_p}{\partial y}\right)$$

The first expression in the right-hand side of Eqn. (23) can be developed

$$\mu_p\left[2\frac{\partial}{\partial x}\left(u'_p\frac{\partial u'_p}{\partial x}\right)+2\frac{\partial}{\partial y}\left(v'_p\frac{\partial v'_p}{\partial y}\right)+\frac{\partial}{\partial x}\left(v'_p\frac{\partial v'_p}{\partial x}+v'_p\frac{\partial u'_p}{\partial y}\right)+\frac{\partial}{\partial y}\left(u'_p\frac{\partial v'_p}{\partial x}+u'_p\frac{\partial u'_p}{\partial y}\right)\right]$$
$$=\mu_p\left[\frac{\partial}{\partial x}\left(u'_p\frac{\partial u'_p}{\partial x}+v'_p\frac{\partial v'_p}{\partial x}\right)+\frac{\partial}{\partial y}\left(u'_p\frac{\partial u'_p}{\partial y}+v'_p\frac{\partial v'_p}{\partial y}\right)\right] \quad (24)$$
$$+\mu_p\left[\frac{\partial}{\partial y}\left(u'_p\frac{\partial v'_p}{\partial x}+v'_p\frac{\partial v'_p}{\partial y}\right)+\frac{\partial}{\partial x}\left(u'_p\frac{\partial u'_p}{\partial x}+v'_p\frac{\partial u'_p}{\partial y}\right)\right]$$
$$=\frac{\partial}{\partial y}\left[\frac{\mu_p}{\sigma_k}\left(\frac{\partial k_p}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[k_p\mu_p\left(\frac{\partial u_p}{\partial y}\right)\right]+\mu_p\left(\frac{\partial u_p}{\partial y}\right)^2$$

Using the Kolmogorov formulation for the turbulent dissipation rate ε_p, the second expression in the right-hand side of Eqn. (23) can be developed

$$\mu_p\left(3\frac{\partial u'_p}{\partial x}\frac{\partial v'_p}{\partial x}+\frac{\partial u'_p}{\partial x}\frac{\partial u'_p}{\partial y}+\frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial y}+3\frac{\partial u'_p}{\partial y}\frac{\partial v'_p}{\partial y}\right)=\rho_p\varepsilon_p \tag{25}$$

The second expression in the right-hand side of Eqn. (16) can be developed

$$\frac{1}{\rho_p}\left(u'_pF'_x+v'_pF'_y\right)=\varepsilon^* \tag{26}$$

Substituting Eqns. (18), (19), (20), (24), (25), and (26) into Eqn. (16), we get the following equation:

$$\begin{aligned}&\rho_p\frac{\partial k_p}{\partial t}+\rho_pu_p\frac{\partial k_p}{\partial x}+\rho_pv_p\frac{\partial k_p}{\partial y}\\&=\frac{\partial}{\partial y}\left[\rho_p\frac{\mu_p}{\sigma_k}\left(\frac{\partial k_p}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[k_p\mu_p\left(\frac{\partial u_p}{\partial y}\right)\right]+\mu_p\left(\frac{\partial u_p}{\partial y}\right)^2-\rho_p\varepsilon_p+\varepsilon^*\end{aligned} \tag{27}$$

Eqn. (27) is the turbulent kinetic energy of the second phase of non-homogeneous two-phase flows.

2.3 Turbulent dissipation rate equation of the second phase ε_p

The equations of turbulent dissipation rate were established by Schraiber et al. as follows:

$$\frac{\partial}{\partial t}\left(\mu_p\frac{\partial u'_p}{\partial y}\frac{\partial u'_p}{\partial y}\right)=2\mu_p\frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u'_p}{\partial t}\right) \tag{28}$$

$$\frac{\partial}{\partial t}\left(\mu_p\frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial x}\right)=2\mu_p\frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial v'_p}{\partial t}\right) \tag{29}$$

Substituting $\partial u'_p/\partial t$, $\partial v'_p/\partial t$ values from Eqns. (12) and (13) into Eqns. (28) and (29), we get the following equations:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\mu_p\frac{\partial u'_p}{\partial y}\frac{\partial u'_p}{\partial y}\right)=2\mu_p\frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left[-\left(u'_p\frac{\partial \bar{u}_p}{\partial x}+v'_p\frac{\partial \bar{u}_p}{\partial y}\right)-\left(\bar{u}_p\frac{\partial u'_p}{\partial x}+\bar{v}_p\frac{\partial u'_p}{\partial y}\right)\right.\\&\left.-\left(u'_p\frac{\partial u'_p}{\partial x}+v'_p\frac{\partial u'_p}{\partial y}\right)+\frac{1}{\rho_p}\left(\frac{\partial \tau'_{pxx}}{\partial x}+\frac{\partial \tau'_{pxy}}{\partial y}\right)+\frac{1}{\rho_p}F'_x\right]\end{aligned} \tag{30}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\mu_p\frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial x}\right)=2\mu_p\frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left[-\left(u'_p\frac{\partial \bar{v}_p}{\partial x}+v'_p\frac{\partial \bar{v}_p}{\partial y}\right)-\left(\bar{u}_p\frac{\partial v'_p}{\partial x}+\bar{v}_p\frac{\partial v'_p}{\partial y}\right)\right.\\&\left.-\left(u'_p\frac{\partial v'_p}{\partial x}+v'_p\frac{\partial v'_p}{\partial y}\right)+\frac{1}{\rho_p}\left(\frac{\partial \tau'_{pxy}}{\partial x}+\frac{\partial \tau'_{pyy}}{\partial y}\right)+\frac{1}{\rho_p}F'_y\right]\end{aligned} \tag{31}$$

Substituting Eqn. (22) into Eqns. (30) and (31), we get the following equations:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\mu_p \frac{\partial u'_p}{\partial y}\frac{\partial u'_p}{\partial y}\right) \\
&= -2\mu_p\left\{u'_p \frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u'_p}{\partial x}\right) + \bar{v}_p \frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u'_p}{\partial y}\right) + v'_p \frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial \bar{u}_p}{\partial y}\right)\right. \\
&+ v'_p \frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u'_p}{\partial y}\right) + \frac{\partial u'_p}{\partial y}\frac{\partial \bar{u}_p}{\partial y}\frac{\partial v'_p}{\partial y} + \frac{\partial u'_p}{\partial y}\frac{\partial u'_p}{\partial y}\frac{\partial \bar{v}_p}{\partial y} + \frac{\partial u'_p}{\partial y}\frac{\partial u'_p}{\partial y}\frac{\partial u'_p}{\partial x} \\
&+ \frac{\partial u'_p}{\partial y}\frac{\partial u'_p}{\partial y}\frac{\partial v'_p}{\partial y} + \frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\frac{\partial\left(u'_p \bar{u}_p\right)}{\partial x} - 2\frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial u'_p}{\partial x}\right)\right] \\
&\left. - \frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial u'_p}{\partial y}\right)\right] - \frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial v'_p}{\partial x}\right)\right] - \frac{1}{\rho_p}\frac{\partial u'_p}{\partial y}\frac{\partial F'_x}{\partial y}\right\}
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\mu_p \frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial x}\right) \\
&= -2\mu_p\left\{u'_p \frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial v'_p}{\partial x}\right) + \bar{u}_p \frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial v'_p}{\partial x}\right) + u'_p \frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial \bar{v}_p}{\partial x}\right)\right. \\
&+ v'_p \frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial v'_p}{\partial y}\right) + \frac{\partial v'_p}{\partial x}\frac{\partial \bar{v}_p}{\partial x}\frac{\partial u'_p}{\partial x} + \frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial x}\frac{\partial \bar{u}_p}{\partial x} + \frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial y} \\
&+ \frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial x}\frac{\partial u'_p}{\partial x} + \frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\frac{\partial\left(v'_p \bar{v}_p\right)}{\partial y} - 2\frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial v'_p}{\partial y}\right)\right] \\
&\left. - \frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial v'_p}{\partial x}\right)\right] - \frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial u'_p}{\partial y}\right)\right] - \frac{1}{\rho_p}\frac{\partial v'_p}{\partial x}\frac{\partial F'_y}{\partial x}\right\}
\end{aligned} \tag{33}
$$

Adding both sides of Eqn. (32) and Eqn. (33) and rearranging, we get the equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\mu_p \frac{\partial u_p^{'}}{\partial y}\frac{\partial u_p^{'}}{\partial y}\right)+\frac{\partial}{\partial t}\left(\mu_p \frac{\partial v_p^{'}}{\partial x}\frac{\partial v_p^{'}}{\partial x}\right)\\
&=-2\mu_p\left\{\left[u_p^{'}\frac{\partial v_p^{'}}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial v_p^{'}}{\partial x}\right)+\bar{u}_p\frac{\partial v_p^{'}}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial v_p^{'}}{\partial x}\right)+u_p^{'}\frac{\partial v_p^{'}}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial \bar{v}_p}{\partial x}\right)\right]\right.\\
&+\left[\bar{v}_p\frac{\partial u_p^{'}}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u_p^{'}}{\partial y}\right)+v_p^{'}\frac{\partial u_p^{'}}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial \bar{u}_p}{\partial y}\right)+v_p^{'}\frac{\partial u_p^{'}}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u_p^{'}}{\partial y}\right)\right]\\
&+\left[u_p^{'}\frac{\partial u_p^{'}}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u_p^{'}}{\partial x}\right)+v_p^{'}\frac{\partial v_p^{'}}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial v_p^{'}}{\partial y}\right)+\frac{\partial u_p^{'}}{\partial y}\frac{\partial}{\partial y}\frac{\partial\left(u_p^{'}\bar{u}_p\right)}{\partial x}+\frac{\partial v_p^{'}}{\partial x}\frac{\partial}{\partial x}\frac{\partial\left(v_p^{'}\bar{v}_p\right)}{\partial y}\right]\\
&+\left[\frac{\partial u_p^{'}}{\partial y}\frac{\partial u_p^{'}}{\partial y}\frac{\partial u_p^{'}}{\partial x}+\frac{\partial u_p^{'}}{\partial y}\frac{\partial u_p^{'}}{\partial y}\frac{\partial v_p^{'}}{\partial y}+\frac{\partial v_p^{'}}{\partial x}\frac{\partial v_p^{'}}{\partial x}\frac{\partial \bar{u}_p}{\partial x}+\frac{\partial v_p^{'}}{\partial x}\frac{\partial v_p^{'}}{\partial x}\frac{\partial v_p^{'}}{\partial y}\right]\\
&+\left[\frac{\partial u_p^{'}}{\partial y}\frac{\partial \bar{u}_p}{\partial y}\frac{\partial v_p^{'}}{\partial y}+\frac{\partial u_p^{'}}{\partial y}\frac{\partial u_p^{'}}{\partial y}\frac{\partial \bar{v}_p}{\partial y}+\frac{\partial v_p^{'}}{\partial x}\frac{\partial \bar{v}_p}{\partial x}\frac{\partial u_p^{'}}{\partial x}+\frac{\partial v_p^{'}}{\partial x}\frac{\partial v_p^{'}}{\partial x}\frac{\partial u_p^{'}}{\partial x}\right]\\
&-\left[2\frac{\partial u_p^{'}}{\partial y}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial u_p^{'}}{\partial x}\right)\right]+2\frac{\partial v_p^{'}}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial v_p^{'}}{\partial y}\right)\right]+\frac{\partial v_p^{'}}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial v_p^{'}}{\partial x}\right)\right]\right.\\
&\left.+\frac{\partial v_p^{'}}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial u_p^{'}}{\partial y}\right)\right]+\frac{\partial u_p^{'}}{\partial y}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial u_p^{'}}{\partial y}\right)\right]+\frac{\partial u_p^{'}}{\partial y}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial v_p^{'}}{\partial x}\right)\right]\right]\\
&\left.-\left[\frac{1}{\rho_p}\frac{\partial u_p^{'}}{\partial y}\frac{\partial F_x^{'}}{\partial y}+\frac{1}{\rho_p}\frac{\partial v_p^{'}}{\partial x}\frac{\partial F_y^{'}}{\partial x}\right]\right\}
\end{aligned}
\tag{34}
$$

The expression in the left-hand side of Eqn. (34) is the turbulent dissipation rate:

$$
\frac{\partial}{\partial t}\left(\mu_p \frac{\partial u_p^{'}}{\partial y}\frac{\partial u_p^{'}}{\partial y}\right)+\frac{\partial}{\partial t}\left(\mu_p \frac{\partial v_p^{'}}{\partial x}\frac{\partial v_p^{'}}{\partial x}\right)=\rho_p\frac{\partial \varepsilon_p}{\partial t} \tag{35}
$$

The first expression in the right-hand side of Eqn. (34) can be developed

$$
2\mu_p\left[\bar{u}_p\frac{\partial v_p^{'}}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial v_p^{'}}{\partial x}\right)+u_p^{'}\frac{\partial v_p^{'}}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial \bar{v}_p}{\partial x}\right)+u_p^{'}\frac{\partial v_p^{'}}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial v_p^{'}}{\partial x}\right)\right]=\rho_p u_p\frac{\partial \varepsilon_p}{\partial x} \tag{36}
$$

The second expression in the right-hand side of Eqn. (34) can be developed

$$
2\mu_p\left[\bar{v}_p\frac{\partial u_p^{'}}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u_p^{'}}{\partial y}\right)+v_p^{'}\frac{\partial u_p^{'}}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial \bar{u}_p}{\partial y}\right)+v_p^{'}\frac{\partial u_p^{'}}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u_p^{'}}{\partial y}\right)\right]=\rho_p v_p\frac{\partial \varepsilon_p}{\partial y} \tag{37}
$$

The third expression in the right-hand side of Eqn. (34) can be developed

$$2\mu_p\left[u'_p\frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u'_p}{\partial x}\right)+v'_p\frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial v'_p}{\partial y}\right)+\frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\frac{\partial\left(u'_p\bar{u}_p\right)}{\partial x}+\frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\frac{\partial\left(v'_p\bar{v}_p\right)}{\partial y}\right] \tag{38}$$
$$=\frac{\partial}{\partial y}\left[\frac{\mu_p}{\sigma_\varepsilon}\left(\frac{\partial\varepsilon_p}{\partial y}\right)\right]$$

The fourth expression in the right-hand side of Eqn. (34) can be developed

$$2\mu_p\left[\frac{\partial u'_p}{\partial y}\frac{\partial u'_p}{\partial y}\frac{\partial u'_p}{\partial x}+\frac{\partial u'_p}{\partial y}\frac{\partial u'_p}{\partial y}\frac{\partial v'_p}{\partial y}+\frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial y}+\frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial x}\frac{\partial u'_p}{\partial x}\right]=C_{\varepsilon 2}\frac{\varepsilon_p^2}{k_p} \tag{39}$$

The fifth expression in the right-hand side of Eqn. (34) can be developed

$$2\mu_p\left[\frac{\partial u'_p}{\partial y}\frac{\partial\bar{u}_p}{\partial y}\frac{\partial v'_p}{\partial y}+\frac{\partial u'_p}{\partial y}\frac{\partial u'_p}{\partial y}\frac{\partial\bar{v}_p}{\partial y}+\frac{\partial v'_p}{\partial x}\frac{\partial\bar{v}_p}{\partial x}\frac{\partial u'_p}{\partial x}+\frac{\partial v'_p}{\partial x}\frac{\partial v'_p}{\partial x}\frac{\partial\bar{u}_p}{\partial x}\right]=\mu_p C_{\varepsilon 1}\frac{\varepsilon_p}{k_p}\left(\frac{\partial u_p}{\partial y}\right)^2 \tag{40}$$

The sixth expression in the right-hand side of Eqn. (34) can be developed

$$2\mu_p\left\{2\frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial u'_p}{\partial x}\right)\right]+2\frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial v'_p}{\partial y}\right)\right]+\frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial v'_p}{\partial x}\right)\right]\right.$$
$$\left.+\frac{\partial v'_p}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial u'_p}{\partial y}\right)\right]+\frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial u'_p}{\partial y}\right)\right]+\frac{\partial u'_p}{\partial y}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial v'_p}{\partial x}\right)\right]\right\} \tag{41}$$
$$=\frac{\partial}{\partial y}\left[\varepsilon_p\frac{\mu_{tp}}{Sc}\left(\frac{\partial u_p}{\partial y}\right)\right]$$

The seventh expression in the right-hand side of Eqn. (34) can be developed

$$2\mu_p\left(\frac{1}{\rho_p}\frac{\partial u'_p}{\partial y}\frac{\partial F'_x}{\partial y}+\frac{1}{\rho_p}\frac{\partial v'_p}{\partial x}\frac{\partial F'_y}{\partial x}\right)=\phi_\varepsilon \tag{42}$$

Substituting Eqns. (35)÷(42) into Eqn. (34), we get the following equation:

$$\rho_p\frac{\partial\varepsilon_p}{\partial t}+\rho_p u_p\frac{\partial\varepsilon_p}{\partial x}+\rho_p v_p\frac{\partial\varepsilon_p}{\partial y}$$
$$=\frac{\partial}{\partial y}\left[\frac{\mu_p}{\sigma_\varepsilon}\left(\frac{\partial\varepsilon_p}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\varepsilon_p\frac{\mu_p}{Sc}\left(\frac{\partial u_p}{\partial y}\right)\right]+\mu_p C_{\varepsilon 1}\frac{\varepsilon_p}{k_p}\left(\frac{\partial u_p}{\partial y}\right)^2+C_{\varepsilon 2}\frac{\varepsilon_p^2}{k_p}+\phi_\varepsilon \tag{43}$$

The experimental coefficients should be (Schraiber 1987)

C_μ=0.09; $C_{\varepsilon 1}$=1.44; $C_{\varepsilon 2}$=1.92; σ_k=1; σ_ε=1.3.

3. CONCLUSIONS

The four-equation turbulence model includes Eqns. (44)÷(47).

$$\rho_g\frac{\partial k_g}{\partial t}+\rho_g u_g\frac{\partial k_g}{\partial x}+\rho_g v_g\frac{\partial k_g}{\partial y}=\frac{\partial}{\partial y}\left[\frac{\mu_g}{\sigma_k}\left(\frac{\partial k_g}{\partial y}\right)\right]+\mu_g\left(\frac{\partial u_g}{\partial y}\right)^2-\rho_g\varepsilon_g-\varepsilon^* \tag{44}$$

$$\rho_p \frac{\partial k_p}{\partial t} + \rho_p u_p \frac{\partial k_p}{\partial x} + \rho_p v_p \frac{\partial k_p}{\partial y} = \frac{\partial}{\partial y}\left[\rho_p \frac{\mu_p}{\sigma_k}\left(\frac{\partial k_p}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[k_p \mu_p \left(\frac{\partial u_p}{\partial y}\right)\right] + \mu_p \left(\frac{\partial u_p}{\partial y}\right)^2 - \rho_p \varepsilon_p + \varepsilon^* \quad (45)$$

$$\rho_g \frac{\partial \varepsilon_g}{\partial t} + \rho_g u_g \frac{\partial \varepsilon_g}{\partial x} + \rho_g v_g \frac{\partial \varepsilon_g}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\mu_g}{\sigma_\varepsilon}\left(\frac{\partial \varepsilon_g}{\partial y}\right)\right] + C_{\varepsilon 1}\mu_g \frac{\varepsilon_g}{k_g}\left(\frac{\partial v_g}{\partial y}\right)^2 - C_{\varepsilon 2}\frac{\varepsilon_g^2}{k_g} - \phi_\varepsilon \quad (46)$$

$$\rho_p \frac{\partial \varepsilon_p}{\partial t} + \rho_p u_p \frac{\partial \varepsilon_p}{\partial x} + \rho_p v_p \frac{\partial \varepsilon_p}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\mu_p}{\sigma_\varepsilon}\left(\frac{\partial \varepsilon_p}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\varepsilon_p \frac{\mu_p}{Sc}\left(\frac{\partial u_p}{\partial y}\right)\right] + \mu_p C_{\varepsilon 1}\frac{\varepsilon_p}{k_p}\left(\frac{\partial u_p}{\partial y}\right)^2 + C_{\varepsilon 2}\frac{\varepsilon_p^2}{k_p} + \phi_\varepsilon \quad (47)$$

Application of the k_g–k_p–ε_g–ε_p turbulent model with additional equations for the turbulent kinetic energy k_p and the turbulent dissipation rate ε_p of the second phase seems to be worthwhile for simulating two-phase non-homogeneous turbulent flows in industrial combustor.

Nomenclature

u_g; v_g	: Velocity components of the gas phase
u_p; v_p	: Velocity components of the second phase
u_g'; v_g'	: Fluctuation of velocity components of the gas phase
u_p'; v_p'	: Fluctuation of velocity components of the second phase
$\bar{u}_g$; $\bar{v}_g$	: Mean of velocity components of the gas phase
$\bar{u}_p$; $\bar{v}_p$	: Mean of velocity components of the second phase
ρ_g;ρ_p	: Density of the gas phase and the second phase
p	: Pressure
ε_g;ε_p	: Turbulent dissipation rate of the gas phase and the second phase
k_g; k_p	: Turbulent kinetic energy of the gas phase and the second phase
μ_g;μ_p	: Dynamic molecular viscosity of the gas phase and the second phase
τ_{gij}; τ_{pij}	: Specific Reynolds stress tensor of the gas phase and the second phase
F_x'; F_y'	: Forces
$C_{\varepsilon 1}$; $C_{\varepsilon 2}$	: Dissipation rate coefficients.
S_{ij}	: Mean strain rate tensor.
σ_k	: Turbulent Prandtl for kinetic energy
σ_ε	: Turbulent Prandtl for dissipation rate
$f_{\varepsilon 1}$; $f_{\varepsilon 2}$	: Boundary functions
μ_L	: Laminar viscosity
μ_T	: Turbulent viscosity
ε^*	: Eddy viscosity
ϕ_ε	: Explicit wall term

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