Lovejoy and Osburn's Overpartitions

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ABSTRACT

In 2008, Lovejoy and Osburn defined the generating function for $\overline{P}(n)$.In 2009, Byungchan Kim defined the generating function for $P_2(n)$.This paper shows how to discuss the generating functions for $\overline{P}(n)$ and $P_2(n)$. Byungchan Kim also defined $P_k(n)$ with increasing relation and overpartition congruences mod 4,8 and 64. In 2006, Berndt found the relation $d_{1,4}(n)-d_{3,4}(n)$ has two values with certain restrictions and various formulae by the common term $\sigma(n)$.This paper shows how to prove the four Theorems about overpartitions modulo 8.These Theorems satisfy the arithmetic properties of the overpartition function modulo 8.

Keywords: Convenience, congruent, modulo 8, prime factorizations, parity

INTRODUCTION

In this paper we give some related definitions of overpartition, $\omega(\lambda)$, $P_k(n)$, $d(n)$, $d_{i,4}(n)$, $\sigma(n)$ and $\chi(n)$. We discuss the generating functions for $\overline{P}(n)$ and $P_2(n)$. We analyze various relations $\overline{P}(n)=\sum_k 2^k p_k(n)$, $\overline{P}(3n+2)\equiv 0 \pmod 4$, $\overline{P}(4n+3)\equiv 0 \pmod 8$,

$$\overline{P}(8n+7)\equiv 0 \pmod{64},$$

$$d_{1,4}(n)-d_{3,4}(n)=\begin{cases}(r_1+1)\ldots(r_k+1), \text{ if } s_i\text{'s are even integers,}\\ 0, \text{ otherwise,}\end{cases}$$

$$\overline{P}(n)\equiv 2d_{1,4}(n)-2d_{3,4}(n)-2\chi(n)-2\sigma(n)+4d(n) \pmod 8,$$

$$\sigma(n)=\left(2^{a+1}-1\right)\prod_i\left(\sum_{m=0}^{r_i}p_i^m\right)\prod_j\left(\sum_{m=0}^{s_j}q_j^m\right), \text{ and } \sigma(n)\equiv\begin{cases}(r_1+1)\ldots(r_k+1) \pmod 4 \text{ if } a=o\\ 3(r_1+1)\ldots(r_k+1) \pmod 4, \text{ otherwise,}\end{cases}$$

respectively. We prove the four Theorems about overpartitions modulo 8 with certain conditions of n.

SOME RELATED DEFINITIONS

Overpartition: An overpartition of n is a partition of n in which the first occurrence of a part may be overlined. Let $\overline{P}(n)$ denote the number of overpartitions of an integer n. For convenience, define

$\overline{P}(0)=1$. For example

n		$\overline{P}(n)$
1 :	$1,\ \bar{1}$	2
2 :	$2,\ \bar{2},\ \ 1+1,\ \bar{1}+1$	4
3 :	$3,\ \bar{3},\ \ 2+1,\ \bar{2}+1,\ 2+\bar{1},\ \bar{2}+\bar{1},\ 1+1+1,\ \bar{1}+1+1$	8
4 :	$4,\ \bar{4},\ \ 3+1,\ \bar{3}+1,\ 3+\bar{1},\ \bar{3}+\bar{1},\ 2+2,\ \bar{2}+2,\ 2+1+1,$ $\bar{2}+1+1,\ 2+\bar{1}+1,\ \bar{2}+\bar{1}+1,\ 1+1+1+1,\ \bar{1}+1+1+1$	14
...	....	...

Similarly we get;

$\overline{P}(5)=24$, $\overline{P}(6)=40$, $\overline{P}(7)=64$,...

$\omega(\lambda)$ **:** An ordinary partition λ, there are $2^{\omega(\lambda)}$ distinct overpartitions, where $\omega(\lambda)$ is the number of distinct parts in λ. For example if λ = 2+1+1; $\omega(\lambda)=2$, there are four overpartitions [2+1+1, $\bar{2}+1+1$, $2+\bar{1}+1$, $\bar{2}+\bar{1}+1$] then. $2^{\omega(\lambda)}=2^2=4$.

$P_k(n)$ [Byungchan Kim(2009)] : The number of partitions of n such that the number of distinct parts is exactly k. For example $P_2(6)=6$ since there are six partitions like 5+1, 4+2, 4+1+1, 3+1+1+1, 2+2+1+1, 2+1+1+1+1.

d(n) : The number of the divisors of n.

$d_{i,4}(n)$ [Alladi (1997)] :The number of the divisors of n which are congruent to i modulo 4.

$\sigma(n)$:The sum of the divisors of n.

$\chi(n)$: The term is defined by $\chi(n)=\begin{cases} 1, & \text{where n is a square of an integer,} \\ 0, & \text{otherwise.} \end{cases}$

For example, $\chi(6)=0, \chi(9)=1, \ldots\ldots$

THE GENERATING FUNCTION

The generating function [Byungchan Kim(2009)] for $\overline{P}(n)$ is given by

$$\prod_{n=1}^{\infty}\frac{(1+x^n)}{(1-x^n)}=\frac{(1+x)(1+x^2)(1+x^3)\ldots\ldots}{(1-x)(1-x^2)(1-x^3)\ldots.}$$

$$= (1+x+x^2+2x^3+2x^4+3x^5+......)\quad(1+x+x^2+3x^3+5x^4+.....)$$

$$= 1+2x+3x^2+8x^3+14x^4+24x^5+40x^6+64x^7+.....$$

$$= \bar{P}(o)+\bar{P}(1)x+\bar{P}(2)x^2+\bar{P}(3)x^3+\bar{P}(4)x^4+.........$$

$$= \sum_{n=0}^{\infty} \bar{P}(n)x^n .$$

The generating function [Byungchan Kim(2009)] for $P_2(n)$ is given by

$$\left(\sum_{k\ge1} \frac{x^k}{1-x^k}\right)^2 - \sum_{k\ge1}\left(\frac{x^k}{1-x^k}\right)^2$$

$$= \left(\frac{x}{1-x}+\frac{x^2}{1-x^2}+....\right)^2 - \left[\left(\frac{x}{1-x}\right)^2+\left(\frac{x^2}{1-x^2}\right)^2+...\right]$$

$$= 2\frac{x}{1-x}.\frac{x^2}{1-x^2}+2.\frac{x}{1-x}.\frac{x^3}{1-x^3}+2.\frac{x}{1-x}.\frac{x^4}{1-x^4}+....$$

$$= 2x^3(1+x+x^2+...)(1+x^2+...)+2x^4(1+x+....)(1+x^3+....)+....$$

$$= 2x^3+4x^4+10x^5+....$$

$$= 2P_2(3)x^3+2P_2(4)x^4+2P_2(5)x^5+...$$

$= \sum_{n\ge1} 2P_2(n)x^n$. For convenience $P_2(1)=0$ and $P_2(2)=0$.

VARIOUS RELATIONS ABOUT OVERPARTITIONS

A) If n = 6, $\bar{P}$ (6) = 40, $P_1(6)=4$ (like : 6, 3+3, 2+2+2, 1+1+1+1+1+1), $P_2(6)=6$, and $P_3(6)=1$

$\therefore$ $2P_1(6)+2^2\ P_2(6)+2^3\ P_3(6)$

= 2.4 + 4.6 + 8.1

= 8 + 24 + 8 = 40 = $\bar{P}(6)$

$\therefore$ $\bar{P}$ (6) = $2P_1(6)+2^2\ P_2(6)+2^3\ P_3(6)$.

So we can write $\bar{P}(n)$ = $\sum_k 2^k P_k(n)$ [Andrews (1967)].

Reducing this modulo 8, we obtain $\bar{P}(n) \equiv 2P_1(n)+2^2\ P_2(n)$ (mod 8), it is seen that $P_1(n)=d(n)$, when d(n) is the number of the divisors of n including 1 and n.

B) We get;

$\bar{P}$ (2)= 4, $\bar{P}$ (5) = 24, ... i.e., $\bar{P}$ (2) = 4 $\equiv$ 0 (mod 4), $\bar{P}$ (3+2)= 24 $\equiv 0$ (mod 4),...

We can conclude that $\overline{P}(3n+2) \equiv 0 \pmod 4$.

C) We get;

$\overline{P}(3) = 8$, $\overline{P}(7) = 64$, ... i.e. $\overline{P}(3) = 8 \equiv 0 \pmod 8$, $\overline{P}(4+3) = 64 \equiv 0 \pmod 8$.

We can conclude that $\overline{P}(4n+3) \equiv 0 \pmod 8$.

D) We get;

$\overline{P}(7) = 64$, $\overline{P}(15)=1408$, ... i.e. $\overline{P}(7)= 64 \equiv 0 \pmod{64}$, $\overline{P}(8+7)=1408 \equiv 0 \pmod{64}$, ...

We can conclude that $\overline{P}(8n+7)) \equiv 0 \pmod{64}$. [Lovejoy et al (2008)]

E) Let $n = 2^a p_1^{r_1} \ldots p_k^{r_k} q_1^{s_1} \ldots . q_l^{s_l}$, ,

If $d_{i,4}(n)$ is the number of the divisors which are congruent to i modulo 4 .

Now if $n = 9 = 3^2 = 3^{s_1}$ when $s_1 = 2$ is the even integer

$\therefore$ $d_{1,4}(9) = 2$, $d_{3,4}(9) = 1$, then $d_{1,4}(9) - d_{3,4}(9) = 2 - 1 = 1$.

Again if $n = 6 = 2.3 = 2^a.3^{s_1}$ when a=1 and $s_1 = 1$

$\therefore$ $d_{1,4}(6) = 1$, $d_{3,4}(6) = 1$,

Then $d_{1,4}(6) - d_{3,4}(6) = 1 - 1 = 0$.

We can conclude that if n has the prime factorization $2^a p_1^{r_1} \ldots\ldots p_k^{r_k} q_1^{s_1} \ldots\ldots . q_l^{s_l}$, where the p_i's are primes congruent to 1 modulo 4 and q_j's are primes congruent to 3 modulo 4, then

$$d_{1,4}(n) - d_{3,4}(n) = \begin{cases} (r_1+1)\ldots.(r_k+1), \text{ if } s_i\text{'s are even integers} \\ 0, \text{ otherwise [Fortin et al (2005)].} \end{cases}$$

F) We get; $d_{1,4}(9) = 2$ (like, the divisors are 1 and 9)

$d_{3,4}(9) = 1$, (like, the divisor is 3)

Now we get; $2\,d_{1,4}(9) - 2\,d_{3,4}(9) - 2\chi(9) - 2\sigma(9) + 4d(n)$

$= 2\times 2 \; -2\times 1 - 2\times 1 - 2\times 13 + 4\times 3$

$= -14 \equiv 2 \pmod 8$, but $\overline{P}(9) = 154 \equiv 2 \pmod 8$.

$\therefore$ $\overline{P}(9) \equiv 2\,d_{1,4}(9) - 2\,d_{3,4}(9) - 2\chi(9) - 2\sigma(9) + 4d(9) \pmod 9$.

We can conclude that, $\overline{P}(n) \equiv 2\,d_{1,4}(n) - 2\,d_{3,4}(n) - 2\chi(n) - 2\sigma(n) + 4d(n) \pmod 8$. [Byungchan Kim(2009)]

G) If $n = 10 = 2.5 = 2^a 5^{r_1}$ where $a = 1$ and $r_1 = 1$

$\therefore \sigma(10) = (2^2 - 1)(5^0 + 5^1)$ but $\sigma(10) = \frac{2^2-1}{2-1}.\frac{5^2-1}{5-1}$

= 3 (1+5) $= \frac{3}{1}.\frac{24}{4}$

= 18 = 18

We can conclude that

$$\sigma(n) = \left(2^{a+1} - 1\right)\prod_i \left(\sum_{m=0}^{r_i} p_i^m\right)\prod_j \left(\sum_{m=0}^{s_j} q_j^m\right)$$. [Andrews (1967)]

H) We get, $n = 9 = 3^2 = 2^a.3^{s_1}$ where a = 0 and $s_1 = 2$

$\sigma(9) = \frac{3^2-1}{3-1} = \frac{26}{2} = 13 \equiv 1$ (mod 4) = $(0+1) \equiv (r_1 + 1)$ (mod 4),

again if $n = 10 = 2.5 = 2^a.5^{r_1}$ where a =1 and $r_1 = 1$

$\sigma(10) = \frac{2^2-1}{2-1}.\frac{5^2-1}{5-1} = 3.\frac{24}{4} = 18 \equiv 2$ (mod 4) $\equiv 6$ (mod 4) = 3.2 = 3(1+1)

$\equiv 3(r_1 + 1)$ (mod 4).

We can write that

$$\sigma(n) \equiv \begin{cases} (r_1+1)....(r_k+1) & \text{(mod 4) if a = 0} \\ 3(r_1+1).....(r_k+1) & \text{(mod 4), otherwise.} \end{cases}$$ [Fortin et al (2005)]

THEOREM

Let n be an integer, then

1) $\overline{P}(n) \equiv 0$ (mod 8), where n is not a square of an odd integer or an even integer and is not a double of a square.
2) $\overline{P}(n) \equiv 2$ (mod 8), if n is a square of an odd integer.
3) $\overline{P}(n) \equiv 4$ (mod 8), if n is a double of a square
4) $\overline{P}(n) \equiv 6$ (mod 8), if n is a square of an integer.

Proof: From above we get;

$$\chi(n) = \begin{cases} 1, & \text{when n is a square of an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

$$\overline{P}(n) \equiv 2\,(d_{1,4}(n) - d_{3,4}(n)) - 2\chi(n) - 2\sigma(n) + 4d(n) \text{ (mod 8), ... (1)}$$

where, $d_{i,4}(n)$ is the number of the divisors which are congruent to i modulo 4.

Now we will consider the three cases according to the parity of r_i and s_j

Case 1: There is an s_j that is odd and r_i is any integer, then

$d_{1,4}(n) - d_{3,4}(n) = 0 \;\; \chi(n) = o$ and $d(n) \equiv 0$ (mod 8).

From (1), we get

$\overline{P}(n) \equiv 2\,(d_{1,4}(n) - d_{3,4}(n)) - 2\chi(n) - 2\sigma(n) + 4d(n)$ (mod 8),

$\equiv$ 0-2×0 -2 $\sigma(n)$ + 0 (mod 8)

or $\overline{P}(n) \equiv -2\,\sigma(n)$ (mod 8) .....(2)

[since if $n = 6 = 2.3 = 2^a.3^{s_1}$ where a = 1 and s_1 in an odd integer, then

$d_{1,4}(6) - d_{3,4}(6) = 0$ and $\chi(6) = 0$ where 6 is not a square,

and d (6) = d (2.3) = (1+1) (1+1) = 4

∴ 4 d (6) = 4.4 = 16 ≡ 0 (mod 8)].

From relation G) we get;

$$\sigma(n) = \left(2^{a+1} - 1\right)\prod_i \left(\sum_{m=0}^{r_i} p_i^m\right)\prod_j \left(\sum_{m=0}^{s_j} q_j^m\right) \text{ [Berndt(2006)]}$$

[since s_j's are odd integers, so $\sum_{m=0}^{s_j} q_j^m \equiv 0$ (mod 4).

∴ $\sigma(n) \equiv o$ (mod 4) and 2 $\sigma(n) \equiv 0$ (mod 8)].

From (2) we can conclude that $\overline{P}(n) \equiv 0$ (mod 8) for such *n*.

Case 2: All s_j's are even and there is an r_i that is odd.

Then, $d_{1,4}(n) - d_{3,4}(n) = (r_1+1)\ldots(r_k+1), \chi(n) = 0$ where n is not a square and 4 $d(n) \equiv 0$ (mod 8).

From (1) we get;

$\overline{P}(n) \equiv 2\,(r_1+1)\ldots(r_k+1) - 2\,\sigma(n)$ (mod 8) ..... (3)

[since if $n = 5.3^2 = 45 \;\; d(45) = d(5.3^2) = (1+1).(2+1) = 2.3 = 6$

∴ 4 d (45) = 4.6 = 24 ≡ 0 (mod 8)]

and $\sigma(n) \equiv (r_1+1)\ldots(r_k+1)$ (mod 4), where s_j's are even r_i's are odd and a =0.

From (3), we can conclude that

$\overline{P}(n) \equiv 0$ (mod 8) , where *n* is not a square of an odd integer or an even integer and is not a double of a square. Hence the Theorem 1 .

[Numerical example 1: If *n* is not a square of an odd integer or an even integer and is not a double of a square. We get; $\overline{P}(3) = 8, \;\; \overline{P}(5) = 24, \ldots$

$\therefore \overline{P}(3) = 8 \equiv 0 (\text{mod } 8)$, $\overline{P}(5) = 24 \equiv 0 (\text{mod } 8), \ldots$

We can conclude that $\overline{P}(n) \equiv 0$ (mod 8),for such *n*.]

Case 3: All the r_i's and s_j's are even.

Suppose that a is o. Then n is a square.

By (1) we deduce that

$$\overline{P}(n) \equiv 2\,(r_1+1)\ldots(r_k+1) - 2 - 2\,\sigma(n) + 4 \pmod 8 \quad \ldots..(4)$$

[since $d_{1,4}(n) - d_{3,4}(n) = (r_1+1)\ldots(r_k+1)$ where s_j's are even and $\chi(n) = 1$ where n is a square of an integer, d(n) $\equiv$ 1 (mod 8) and also, $\sigma(n) \equiv (r_1+1)\ldots(r_k+1) \pmod 4$ where r_i's and s_j's are even and also a = 0]

From (4) we get; $\overline{P}(n) \equiv 2\,(r_1+1)\ldots(r_k+1) + 2 - 2\,(r_1+1)\ldots(r_k+1) \pmod 8$.

$\therefore \quad \overline{P}(n) \equiv 2$ (mod 8), when n is a square of an odd integer. Hence the Theorem 2 .

[Numerical example 2: If *n* is not a square of an odd integer,

We get; $\overline{P}(1) = 2,\ \ \overline{P}(9) = 154, \ldots$

$\therefore \quad \overline{P}(1) = 2 \equiv 2 (\mathrm{mod}\, 8)\ ,\ \ \overline{P}(9) = 154 \equiv 2 (\mathrm{mod}\, 8), \ldots$

We can conclude that $\overline{P}(n) \equiv 2$ (mod 8),for such *n*.]

Suppose that a is odd. Then n is a double of square.

From (1) we get;

$$\overline{P}(n) \equiv 2\,(r_1+1)\ldots(r_k+1) - 2\,\sigma(n) \pmod 8. \text{ [Berndt(2006)]}$$

[since $d_{1,4}(n) - d_{3,4}(n) = (r_1+1)\ldots(r_k+1)$ where r_i's and s_j's are even integers.

χ (n) =o, where n is not a square of an integer.

If $n = 2.3^2.5^2$

$\therefore \quad$ d (n) = (1+1). (2+1). (2+1)=18

$\therefore \quad$ 4d (n) = 4.18=72$\equiv$0 (mod 8)]

$\therefore \quad \overline{P}(n) \equiv 2\,(r_1+1)\ldots(r_k+1) - 2.3\,(r_1+1)\ldots(r_k+1) \pmod 8$

[since $\sigma(n) \equiv 3\,(r_1+1)\ldots(r_k+1) \pmod 4$, where a is not zero]

$$\overline{P}(n) \equiv -4\,(r_1+1)\ldots(r_k+1) \pmod 8$$

$$\equiv 4\,(r_1+1)\ldots(r_k+1) \pmod 8$$

$$\equiv 4 \pmod 8.$$

[since r_i's and s_j's are even integers so, $(r_1+1)\ldots(r_k+1) \equiv 1$ (mod 8)] [Fortin et al (2005]

$\therefore \quad \overline{P}(n) \equiv 4$ (mod 8), when n is a double of a square. Hence the Theorem 3 .

[Numerical example 3: If *n* is a double of a square. We get; $\overline{P}(2) = 4,\ \ \overline{P}(8) = 100, \ldots$

$\therefore\ \overline{P}(2) = 4 \equiv 4 (\mathrm{mod}\, 8)\ ,\ \ \overline{P}(2.4) = 100 \equiv 4 (\mathrm{mod}\, 8), \ldots$

We can conclude that $\overline{P}(n) \equiv 4 (\mathrm{mod}\, 8)$,for such *n*.]

Suppose that a is even. Then n is a square of an even integer.

From (1) we get; $\overline{P}(n) \equiv 2\,(r_1+1)\ldots(r_k+1) - 2 - 2\,\sigma(n) + 4 \pmod 8$

[since $d_{1,4}(n)-d_{3,4}(n)=(r_1+1)\ldots(r_k+1)$ where r_i's and s_j's are even integers, χ (n) =1, where n is a square of an integer and $d(n)\equiv 1 \pmod 8$)].

or $\overline{P}(n) \equiv 2\ (r_1+1)\ldots(r_k+1)+2\text{-}2.3\,(r_1+1)\ldots(r_k+1) \pmod 8$

[since $\sigma(n)\equiv 3\ (r_1+1)\ldots(r_k+1) \pmod 4$, where a $\neq 0$]

or $\overline{P}(n) \equiv -4\ (r_1+1)\ldots(r_k+1)+2 \pmod 8$

$\equiv 4\ (r_1+1)\ldots(r_k+1)+2 \pmod 8$

$\equiv 4.1\ +2 \pmod 8$

[since r_i's and s_j's are even integers so $(r_1+1)\ldots(r_k+1)\equiv 1 \pmod 8$].

$\therefore$ $\overline{P}(n) \equiv 6 \pmod 8$, when n is a square of an even integer. Hence the Theorem 4 .

[Numerical example 4: If *n* is a square of an even integer. We get; $\overline{P}(4)=14, \ldots$

$\therefore\ \overline{P}(4)=14\equiv 6(\bmod 8), \ldots$

We can conclude that $\overline{P}(n)\equiv 6(\bmod 8)$,for such *n*.]

CONCLUSION

In this study we have analyzed various relations $\overline{P}(n)=\sum_k 2^k p_k(n)$, $\overline{P}(3n+2)\equiv 0$ (mod 4),

$\overline{P}(4n+3)\equiv 0 \pmod 8$, $\overline{P}(8n+7)\equiv 0 \pmod{64}$,

$$d_{1,4}(n)-d_{3,4}(n)=\begin{cases}(r_1+1)\ldots(r_k+1), \text{ if } s_i\text{'s are even integers,}\\ 0, \text{ otherwise}\end{cases},$$

$\overline{P}(n)\equiv 2d_{1,4}(n)-2d_{3,4}(n)-2\chi(n)-2\sigma(n)+4d(n) \pmod 8$,

$$\sigma(n)=\left(2^{a+1}-1\right)\prod_i\left(\sum_{m=0}^{r_i} p_i^m\right)\prod_j\left(\sum_{m=0}^{s_j} q_j^m\right), \text{ and } \sigma(n)\equiv\begin{cases}(r_1+1)\ldots(r_k+1)(\bmod 4) \text{ if } a=o\\ 3(r_1+1)\ldots(r_k+1) \pmod 4, \text{ otherwise,}\end{cases}$$

respectively with the help of numerical examples. We have verified the four Theorems about overpartitions modulo 8 with numerical examples.

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REFERENCES

Ahmad, M. (2013). Homogeneous Number System and Reciprocal Symmetric Algebra. *Asian Journal Of Applied Science And Engineering,* 2(1), 92-99. Retrieved from http://journals.abc.us.org/index.php/ajase/article/view/2.10%28M%29

Ahmad, M., & Talukder, M. (2013). Correspondence between Reciprocity and Discreteness. *Asian Journal Of Applied Science And Engineering*, *2*(1), 16-19. Retrieved fromhttp://journals.abc.us.org/index.php/ajase/article/view/2.2M%26G

Alladi K., A fundamental invariant in the theory of partitions, in: Topics in Number Theory (University Park, PA, 1997), Kluwer Acad, Pubi, Dordrecht, 1999, pp.101-113

Andrews G.E., Enumerative proofs of certain q-identities, Glasg. Math.J.B(1967)33-40

Berndt B.C., Number Theory in the Spirit of Ramanujan, American Mathematical Society, Providence, RI, 2006.

Byungchan Kim, A short note on the overpartition function, Discrete Mathematics 309,(2009), 2528-2532.

Das S, Mohajan HK. The Number of Vector Partitions of n (Counted According to the weight) with the Crank m International Journal of Reciprocal Symmetry and Theoretical Physics. 2014;1(2):91-105.

Das, S. (2014). Congruence Properties of Andrews'SPT- Function. *ABC Journal Of Advanced Research*, *3*(2), 47-56. Retrieved from http://journals.abc.us.org/index.php/abcjar/article/view/6.5

Duviryak A. Bound States in the Compactified Gravity International Journal of Reciprocal Symmetry and Theoretical Physics. 2014;1(2):80-90.

Fortin J.F.,Jacob P., Mathieu P., Jagged partitions, Ramanujan J. 10 (2005) 215-235.

Lovejoy J. and Osburn R. ,Rank differences for overpartitions, Q.J.Math. 59(2), 257-273,2008.

Mohajan HK. Gravitational Collapse of a Massive Star and Black Hole Formation International Journal of Reciprocal Symmetry and Theoretical Physics. 2014;1(2):125-140.

Mohajan, H. (2014). Upper Limit of the Age of the Universe with Cosmological Constant. *International Journal Of Reciprocal Symmetry And Theoretical Physics*, *1*(1), 43-68. doi:10.15590/ijrstp/2014/v1i1/53723

Talukder, M., & Ahmad, M. (2013). Wave Particle Dualism for Both Matter and Wave and Non-Einsteinian View of Relativity. *Asian Journal Of Applied Science And Engineering*, *2*(1), 80-91. Retrieved fromhttp://journals.abc.us.org/index.php/ajase/article/view/2.9G%26M

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